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EXOGENOUS TECHNOLOGICAL PROGRESS IN THE NEOCLASSICAL RAMSEY–CASS–KOOPMANS MODEL: EFFECTS ON ECONOMIC GROWTH AND WELFARE

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Summary. *The article assesses the impact of changes in the rate of exogenous Harrod-neutral technological progress on economic growth and aggregate discounted utility (welfare) in the EU and the USA within the Ramsey–Cass–Koopmans model. The economy is represented by a public and a private sector producing a single good. The government plays a passive role: it levies taxes (output tax and consumption tax), does not affect technology, does not optimize expenditures, neither borrows nor issues debt. The private sector is represented by a representative household that maximizes welfare according to CRRA preferences. Based on reasonable benchmark values of structural parameters for the EU and the US, a numerical experiment was conducted: trajectories for 70 years were modeled for the baseline and scenarios with an increase in the rate of technological progress. This made it possible to estimate the dynamic paths and long-run (steady-state) sensitivities of capital stock, consumption, and output per worker to changes in key parameters. The results show that technological progress is a key factor in long-term growth in modern Western economies. In particular, an increase in the rate of technological progress by 10% relative to the baseline scenario results in an increase in consumption and output per worker by 8–12% and in capital intensity by 7–11% in 70 years (lower bound for the EU, upper bound for the US). It is shown that the impact of technological progress is nonlinear (growing in time): the growth of capital intensity as a function of the rate of technological progress is convex and increases with the increase in the rate of technological progress. Since the Ramsey–Cass–Koopmans model optimizes the growth trajectory according to the utilitarian social welfare function, the impact of an increase in the rate of technological progress on welfare growth according to the Lucasian equivalent-consumption approach at a finite horizon is analyzed. It is shown that a 10% increase in the rate of technical progress is equivalent to a constant increase in consumption in the baseline scenario of about 3.49% for the EU and 5.28% for the US annually for the first 70 years, which confirms the positive impact of exogenous technological progress on welfare.*
Key words: *technological progress, economic modeling, welfare, economic growth.*

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ЕКЗОГЕННИЙ ТЕХНІЧНИЙ ПРОГРЕС У НЕОКЛАСИЧНІЙ МОДЕЛІ РЕМЗІ–КАССА–КУПМАНСА: ЕФЕКТИ ДЛЯ ЕКОНОМІЧНОГО ЗРОСТАННЯ І ДОБРОБУТУ

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Резюме. *Оцінено вплив зміни темпу екзогенного Харрод-нейтрального технічного прогресу на рівні показників економічного зростання та агреговану дисконтовану корисність (добробут) у країнах ЄС і США в межах моделі Ремзі–Касса–Купманса. Економіка представлена у вигляді державного і приватного секторів з єдиним благом. Роль держави пасивна: вона лише оподатковує (податок на випуск і податок на споживання), не впливає на технологію, не оптимізує витрати, не здійснює запозичень і не проводить емісії. Приватний сектор представлений репрезентативним домогосподарством, яке максимізує добробут за CRRA-уподобаннями. На основі обґрунтованих еталонних значень структурних параметрів для ЄС і США проведено числовий експеримент: змодельовано траєкторії на 70 років для базового сценарію і сценаріїв з підвищенням темпу технічного прогресу. Це дозволило оцінити часові профілі й довготривалу (стаціонарну) чутливість рівнів капіталоозброєності, споживання та випуску на* *одного працівника до змін ключових параметрів. Отримані результати показують, що технічний прогрес є ключовим чинником довготривалого зростання в сучасних західних економіках. Зокрема, підвищення темпу технічного прогресу на 10% відносно базового сценарію дає через 70 років приріст споживання і випуску на одного працівника на 8–12%, а капіталоозброєності – на 7–11% (нижня межа – для ЄС, верхня – для США). Показано, що вплив технічного прогресу має нелінійний (зростаючий у часі) характер: приріст капіталоозброєності як функція темпу технічного прогресу є опуклим і посилюється зі збільшенням темпу технічного прогресу. Оскільки модель Ремзі–Касса–Купманса оптимізує траєкторію зростання за утилітарною соціальною функцією добробуту, проаналізовано вплив підвищення темпу технічного прогресу на зростання добробуту за методом Р. Лукаса на скінченному горизонті. Показано, що підвищення темпу технічного прогресу на 10% еквівалентне постійній надбавці до споживання у базовому сценарії близько 3,49% для ЄС і 5,28% для США щороку протягом перших 70 років, що підтверджує позитивний вплив екзогенного технічного прогресу на добробут.*

Ключові слова: *технічний прогрес, економічне моделювання, добробут, економічне зростання.*

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Problem statement. In the neoclassical tradition, technological progress is understood primarily as an exogenous process independent of the behavior of economic agents and is regarded as the main driver of economic growth and development. The sources of technological progress within this paradigm are not explained (treated as a «black box»), which reflects a certain abstraction from innovation in the Schumpeterian sense. The canonical models of the neoclassical theory of the mid-twentieth century were those of Solow–Swan (1956–1957) and Ramsey–Cass–Koopmans (1965).

For decades, studies of economic growth maintained the assumption of exogeneity of technological progress; only in the late 1980s did economic science return to the Schumpeterian approach, which considers innovation as an endogenous factor of growth. Despite the assumption of exogenous knowledge and technology, these models remain powerful tools for macroeconomic analysis due to their simplicity and analytical transparency, and they provide a solid methodological basis for both theoretical and empirical research.

In the Solow–Swan model, the saving rate (s) is fixed exogenously at a constant level, which makes the framework tractable and easy to use. However, it has the disadvantage of ignoring the behavior of households and their motives to optimize utility from consumption throughout the growth path. From this point of view, the Ramsey–Cass–Koopmans model is theoretically more complex than the basic Solow–Swan model: by introducing optimizing households with intertemporal utility, it permits analysis of the optimal paths of consumption and capital accumulation under exogenous technological progress.

Despite the development of models that endogenize technological progress, the current stage of digital transformation has renewed interest in exogenous approaches. In particular, since the impact of artificial intelligence is difficult to formalize, it can be considered an exogenous factor within simplistic macro models – even though the development of artificial intelligence itself is undoubtedly the result of endogenous efforts by many stakeholders. Since the micro-mechanisms of endogenous growth with the impact of artificial intelligence still need to be thoroughly studied, exogenous models allow for quick and transparent predictions. That is why the theoretical interest in models with exogenous technological progress, in particular the Ramsey–Cass–Koopmans model, is both timely and scientifically sound.

Analysis of recent research and publications. In the modern scientific literature, the canonical Ramsey–Cass–Koopmans model is used primarily as a workhorse for studying the dynamics of economic growth, being extended in various directions and embedded within richer macroeconomic frameworks. The main interest is usually in comparing tax policies from a welfare perspective, analyzing the effects of shocks and transitions (demographic, investment), as well as characterizing optimal growth paths and the conditions of decentralized equilibria.

In [1], the author investigates new properties of capital and consumption dynamics in the canonical Ramsey–Cass–Koopmans setup, examines behavior at low capital endowment, and interprets the saddle path. Thus, the emphasis is placed on the growth dynamics caused by capital accumulation rather than productivity gains from technological progress.

In [2], technology is treated as a random productivity shock. The author notes that the inclusion of exogenous productivity growth creates methodological difficulties: the solution ceases to be strictly Markovian. Only when the growth rate is constant can the Markov structure be retained and the proposed method applied.

In [3], the production function is modified so that, in addition to private capital and effective labor, it includes public spending and technological innovation. Effective labor is treated as a combination of quantity and quality of labor, where quality is determined by the level of technological progress. A separate production function is introduced for the technological process, which depends on the previous stock of knowledge and public R&D expenditures. Thus, technological progress is no longer considered exclusively as an exogenous factor, but becomes endogenized through fiscal policy and government priorities in the area of technological innovation.

In [4], technological progress within the Ramsey–Cass–Koopmans model is described as endogenous, capital- and labor-saving. The authors extend the standard model by allowing firms to invest in innovations that increase capital or labor productivity. The knowledge arising from this process accumulates and supports long-term economic growth.

However, the impact of exogenous technological progress on optimal growth trajectories and overall social welfare remains insufficiently studied relative to the empirical evidence. The Ramsey–Cass–Koopmans model can be used to provide a theoretical basis for and to address applied economic-policy questions, in particular assessing the need for public support for R&D, education, and science.

Purpose of the study is to assess how changes in the rate of Harrod-neutral technological progress affect economic growth and aggregate discounted utility (welfare) in the EU and US using the Ramsey–Cass–Koopmans model.

Statement of the task. To formalize the Ramsey-Cass-Koopmans model in the Cobb–Douglas specification with Harrod-neutral technological progress in per-effective-worker variables; to derive the first-order conditions (Euler equation) and the transversality condition; to analytically obtain steady-state values of capital stock, consumption and output per effective worker and to define a linearization of the system around the steady state, construct the stable manifold, and specify a consumption rule along it; to calibrate structural parameters for the EU and the US; to simulate trajectories for 70 years for the baseline scenario and scenarios with changes in the rate of technological progress; to assess the long-run sensitivities of levels to changes in structural parameters; to compute consumption-equivalent welfare (Lucasian approach) at a finite horizon; to interpret the results and draw conclusions.

Statements of the main material of the study. *Mathematical formulation of the model.* Consider an economy in which the public and private sectors operate. The private sector is represented by a representative household that seeks to maximize its welfare with a utility function exhibiting constant relative risk aversion (parameter $\sigma > 0$) [5, p. 183]. According to the concept of a representative agent, aggregate demand is described as if all decisions on consumption, saving, and labor supply are made by a single hypothetical household subject to a single budget constraint. The main analytical advantage of this assumption is that, instead of modeling the interaction of many heterogeneous agents, the market equilibrium can be represented by a single optimization problem [6, p. 149].

The mathematical formulation of the model is generally consistent with [5, pp. 183–200], except that Harrod-neutral technological progress is considered here, and all variables are expressed in per-effective-labor terms. Under these assumptions, according to Uzawa's theorem, long-run sustainable growth is possible, which allows the analysis of the impact of technological progress on economic development.

A representative agent uses labor, capital, and technology to produce a single good that can be consumed or accumulated in the form of physical capital. The production function with constant returns to scale captures labor-augmenting (Harrod-neutral) technological progress:

$$Y_t = F(K_t, A_t N_t),\ \ \tilde{y}_t = f(\tilde{k}_t), \tag{1}$$

where Y_t is aggregate output; N_t is the population (assumed $L_t = N_t$); A_t is the level of technology; $\tilde{k}_t = K_t / A_t N_t$ is the capital stock per effective worker; $\tilde{y}_t$ is output per effective worker. The index t represents a discrete time period.

The role of the state is passive: it does not optimize expenditures, does not influence technology, and only taxes. The public sector finances its expenditures through taxes on output (τ_t^y) and consumption (τ_t^c). Public spending does not directly affect either the production function or the utility function of the representative agent – it does not return to the economy, i.e., it does not create public utility (to use the metaphor [5, p. 183], «thrown into the sea»). Accordingly, the resulting competitive-equilibrium allocation is not efficient.

The model assumes that the government does not borrow or issue money. Under these conditions, the government's budget constraint, expressed in per-effective-worker terms, is determined by the levels of output and consumption and is as follows:

$$\tilde{g}_t = \tau_t^y f(\tilde{k}_t) + \tau_t^c \tilde{c}_t, t = 0,1,2, \dots \tag{2}$$

where $\tilde{c}_t$ is consumption per effective worker.

The representative agent's budget constraint takes into account that the government withdraws part of income through taxes on consumption and output. Structural parameters are also introduced into the model: population growth rate n; Harrod-neutral technological progress rate γ, and depreciation rate δ. The budget constraint in per-effective-worker terms is then given by:

$$(1 + \tau_t^c)\tilde{c}_t + (1 + n)(1 + \gamma)\tilde{k}_{t+1} - (1 - \delta)\tilde{k}_t = (1 - \tau_t^y) f(\tilde{k}_t). \tag{3}$$

The main idea of the Ramsey–Cass–Koopmans model is that a representative household maximizes intertemporal utility from consumption:

$$\max_{\{\tilde{c}_t, \tilde{k}_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(\tilde{c}_t), \tag{4}$$

where $\beta \in (0,1)$ is the discount factor, and the initial value of $\tilde{k}_0$ is set exogenously.

Similar optimization problems in discrete time are solved using first-order conditions (Euler equation) and the transversality condition that guarantee optimal intertemporal choices. Substituting the budget constraint (3), we obtain the Lagrangian:

$$L = \sum_{t=0}^{\infty} \beta^t \left[U(\tilde{c}_t) + \lambda_t \left((1 - \tau_t^y) f(\tilde{k}_t) - (1 + \tau_t^c)\tilde{c}_t - (1 + n)(1 + \gamma)\tilde{k}_{t+1} + (1 - \delta)\tilde{k}_t \right) \right]. \tag{5}$$

First-order conditions:

$$\frac{\partial L}{\partial \tilde{c}_t}:\ \ \beta^t [U'(\tilde{c}_t) - \lambda_t (1 + \tau_t^c)] = 0 \ \Rightarrow\ \lambda_t = \frac{U'(\tilde{c}_t)}{1 + \tau_t^c}; \tag{6}$$

$$\frac{\partial L}{\partial \tilde{k}_{t+1}}:\ \ -(1 + n)(1 + \gamma)\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} \left[(1 - \tau_{t+1}^y) f'(\tilde{k}_{t+1}) + 1 - \delta \right] = 0. \tag{7}$$

A key condition in problems with an infinite horizon is the transversality condition, which ensures that an agent cannot finance unbounded consumption by accumulating capital without bound. It requires that the discounted shadow value of capital per effective worker tends to zero:

$$\lim_{t\to\infty} \beta^t \lambda_t \tilde{k}_{t+1} = 0. \quad (8)$$

Assume that the production function has constant returns to scale and is Cobb–Douglas with labor-augmenting (Harrod-neutral) technology:

$$F(K_t, A_t N_t) = \varphi K_t^{\alpha}(A_t N_t)^{1-\alpha} \Rightarrow \tilde{y}_t = f(\tilde{k}_t) = \varphi \tilde{k}_t^{\alpha}, f'(\tilde{k}_t) = \varphi\alpha\tilde{k}^{\alpha-1},$$

where $\varphi > 0$ is the level of total factor productivity (TFP).

The utility function exhibits constant relative risk aversion (CRRA): $U(\tilde{c}_t) = \frac{\tilde{c}_t^{1-\sigma}-1}{1-\sigma}$.

Under these assumptions, the Lagrangian takes the form:

$$L = \sum_{t=0}^{\infty} \beta^t \left[\frac{\tilde{c}_t^{1-\sigma} - 1}{1-\sigma} + \lambda_t\Big(\big(1-\tau_t^y\big)\varphi\tilde{k}_t^{\alpha} - (1+\tau_t^c)\tilde{c}_t - (1+n)(1+\gamma)\tilde{k}_{t+1} + (1-\delta)\tilde{k}_t\Big)\right] \quad (9)$$

First-order conditions:

$$\frac{\partial L}{\partial \tilde{c}_t}:\ \beta^t[\tilde{c}_t^{-\sigma} - \lambda_t(1+\tau_t^c)] = 0 \ \Rightarrow\ \lambda_t = \frac{\tilde{c}_t^{-\sigma}}{1+\tau_t^c}; \quad (10)$$

$$\frac{\partial L}{\partial \tilde{k}_{t+1}}:\ -(1+n)(1+\gamma)\beta^t\lambda_t + \beta^{t+1}\lambda_{t+1}\big[\big(1-\tau_{t+1}^y\big)\varphi\alpha\tilde{k}_{t+1}^{\alpha-1} + 1 - \delta\big] = 0. \quad (11)$$

The transversality condition is similar to condition (8), but it can be written in a more intuitive form in which the Lagrange multiplier λ_t is expressed in terms of economically interpretable variables. It states that the present value of residual capital, discounted at the market return, should tend to zero:

$$\lim_{t\to\infty} \frac{[(1+n)(1+\gamma)]^t}{\prod_{s=1}^{t}[(1-\tau_s^y)\varphi\alpha\tilde{k}_s^{\alpha-1}+1-\delta]}\tilde{k}_{t+1} = 0. \quad (12)$$

The dynamics of consumption is determined by the Keynes–Ramsey equation, a necessary condition for optimality in the Ramsey–Cass–Koopmans model, which relates the growth rate of consumption to intertemporal preferences and the after-tax return on capital:

$$\tilde{c}_{t+1} = \left\{\frac{\beta}{(1+n)(1+\gamma)}\frac{1+\tau_t^c}{1+\tau_{t+1}^c}\big[\big(1-\tau_{t+1}^y\big)\varphi\alpha\tilde{k}_{t+1}^{\alpha-1} + 1 - \delta\big]\right\}^{1/\sigma}\tilde{c}_t. \quad (13)$$

The task of the representative agent is to find sequences $\{\tilde{c}_t, \tilde{k}_{t+1}\}_{t=0}^{\infty}$, defined over an infinite horizon, that maximize household utility, ensure clearing of the single-good market, and satisfy the government's budget constraint in each period (2). However, competitive allocation is not Pareto efficient due to distortionary taxation (see [5, p. 150]).

Taken together the agent's budget constraint (3), the Keynes–Ramsey equation (13), and the transversality condition (12), we obtain a nonlinear system of difference equations describing the optimal trajectories of $(\tilde{k}_t, \tilde{c}_t)$ (in units of effective labor) for a given initial value of $\tilde{k}_0$. Convergence to the steady state is not guaranteed for any $\tilde{c}_0$, since the model is saddle-stable: the economy converges to the steady state only when the initial consumption $\tilde{c}_0$ is chosen on a single saddle path.

To analyze the local stability properties, we linearize the system near the steady state. For a technology with diminishing marginal returns to capital, the long-run growth rate of the variables per effective worker is zero. Therefore, in the steady state: $\tilde{c}_t = \tilde{c}_{ss}$, $\tilde{k}_t = \tilde{k}_{ss}$, $\tilde{y}_t = \tilde{y}_{ss}$, for all t.

For simplicity, the government levies constant tax rates on output and consumption: $\tau_t^y = \tau^y$, $\tau_t^c = \tau^c$, $\forall t$ [5, c. 186]. The tax rates are dimensionless fractions, so expressing variables in per-effective-labor terms does not change them (only the levels of the variables change). If a lump-sum tax were introduced, it would have to be applied per person rather than per effective worker.

Substituting these assumptions into the Keynes–Ramsey equation (13), we obtain that a consumption tax does not affect long-run capital accumulation. Then the steady-state level of capital is determined from the equation:

$$1 = \frac{\beta}{(1+n)(1+\gamma)}\left[(1-\tau^y)\varphi\alpha\tilde{k}_{ss}^{\alpha-1} + 1 - \delta\right]$$

and is equal to

$$\tilde{k}_{ss} = \left[\frac{(1-\tau^y)\varphi\alpha}{(1+n)(1+\gamma)\beta^{-1}-(1-\delta)}\right]^{1/(1-\alpha)}, \qquad (14)$$

provided that the steady state exists when $\frac{(1+n)(1+\gamma)}{\beta} > 1 - \delta$.

The steady-state level of consumption follows from the steady-state resource constraint:

$$(1+\tau^c)\tilde{c}_{ss} = (1-\tau^y)\varphi\tilde{k}_{ss}^{\alpha} - [(1+n)(1+\gamma) - (1-\delta)]\tilde{k}_{ss} = (1-\tau^y)\varphi\tilde{k}_{ss}^{\alpha} - (n+\gamma+n\gamma+\delta)\tilde{k}_{ss}. \qquad (15)$$

Obviously, government spending also tends to its steady state:

$$\tilde{g}_{ss} = \tau^c\tilde{c}_{ss} + \tau^y\varphi\tilde{k}_{ss}^{\alpha}. \qquad (16)$$

We now apply a first-order linear approximation (according to Taylor's theorem) around the steady state. To do this, we write the budget constraint in the form:

$$\tilde{k}_{t+1} - \frac{1-\tau^y}{(1+n)(1+\gamma)}\varphi\tilde{k}_t^{\alpha} - \frac{1-\delta}{(1+n)(1+\gamma)}\tilde{k}_t + \frac{1+\tau^c}{(1+n)(1+\gamma)}\tilde{c}_t = 0. \qquad (17)$$

Then we can think of it as a function of $G\left(\tilde{k}_{t+1}, \tilde{c}_{t+1}, \tilde{k}_t, \tilde{c}_t\right) = 0$. The linear approximation around $\left(\tilde{k}_{ss}, \tilde{c}_{ss}\right)$ takes the form:

$$\left(\tilde{k}_{t+1} - \tilde{k}_{ss}\right) - \frac{1}{(1+n)(1+\gamma)}\left[(1-\tau^y)\varphi\alpha\tilde{k}_{ss}^{\alpha-1} + 1 - \delta\right]\left(\tilde{k}_t - \tilde{k}_{ss}\right) + \frac{1+\tau^c}{(1+n)(1+\gamma)}(\tilde{c}_t - \tilde{c}_{ss}) = 0. \qquad (18)$$

From the steady-state condition for $\tilde{k}_{ss}$ from (13) at $\tilde{c}_{t+1} = \tilde{c}_t$, it is easy to see that

$$\frac{\beta}{(1+n)(1+\gamma)}\left[(1-\tau^y)\varphi\alpha\tilde{k}_{ss}^{\alpha-1} + 1 - \delta\right] = 1.$$

Hence, equation (17) takes the form:

$$\tilde{k}_{t+1} - \tilde{k}_{ss} = \frac{1}{\beta}\left(\tilde{k}_t - \tilde{k}_{ss}\right) - \frac{1+\tau^c}{(1+n)(1+\gamma)}(\tilde{c}_t - \tilde{c}_{ss}). \qquad (19)$$

The optimality condition for the representative agent's problem (13) is expressed as the function $F\left(\tilde{k}_{t+1}, \tilde{c}_{t+1}, \tilde{k}_t, \tilde{c}_t\right) = 0$. The linear approximation around the steady state is given by:

$$(\tilde{c}_{t+1} - c_{ss}) = \frac{1}{\sigma} \Psi_{ss}^{-1} \left[-\frac{\beta}{(1+n)(1+\gamma)}(1-\tau^y)\varphi\alpha(\alpha-1)\tilde{k}_{ss}^{\alpha-2}\right] \tilde{c}_{ss}\left(\tilde{k}_{t+1} - \tilde{k}_{ss}\right) + \Psi_{ss}^{1/\sigma}(\tilde{c}_t - \tilde{c}_{ss}), \tag{20}$$

where

$$\Psi_{ss} = \frac{\beta}{(1+n)(1+\gamma)}\left[(1-\tau^y)\varphi\alpha\tilde{k}_{ss}^{\alpha-1} + 1 - \delta\right] = 1 \tag{21}$$

Thus, we obtain:

$$\tilde{c}_t - \tilde{c}_{ss} = (\tilde{c}_{t+1} - c_{ss}) + \frac{1}{\sigma(1+n)(1+\gamma)}\beta(1-\tau^y)\varphi\alpha(1-\alpha)\tilde{k}_{ss}^{\alpha-2}\tilde{c}_{ss}\left(\tilde{k}_{t+1} - \tilde{k}_{ss}\right). \tag{22}$$

Continuing the methodological approach proposed in [5, p. 187], both approximations – the budget constraint (19) and Keynes–Ramsey condition (20) – can be represented as a matrix system:

$$\begin{pmatrix} 1 & 0 \\ M & 1 \end{pmatrix} \begin{pmatrix} \tilde{k}_{t+1} - \tilde{k}_{ss} \\ \tilde{c}_{t+1} - \tilde{c}_{ss} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} & -\frac{1+\tau^c}{(1+n)(1+\gamma)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{k}_t - \tilde{k}_{ss} \\ \tilde{c}_t - \tilde{c}_{ss} \end{pmatrix}, \tag{23}$$

where

$$M = -\frac{1}{\sigma}\frac{\beta}{(1+n)(1+\gamma)}(1-\tau^y)\varphi\alpha(\alpha-1)\tilde{k}_{ss}^{\alpha-2}\tilde{c}_{ss} > 0 \tag{24}$$

Denoting $\tilde{x}_t = \begin{pmatrix} \tilde{k}_t - \tilde{k}_{ss} \\ \tilde{c}_t - \tilde{c}_{ss} \end{pmatrix}$, we obtain a compact form:

$$B_0 \tilde{x}_{t+1} = B_1 \tilde{x}_t,$$

where the coefficient matrices take the form:

$$B_0 = \begin{pmatrix} 1 & 0 \\ M & 1 \end{pmatrix}, .B_1 = \begin{pmatrix} \frac{1}{\beta} & -\frac{1+\tau^c}{(1+n)(1+\gamma)} \\ 0 & 1 \end{pmatrix}$$

Since the matrix B_0 is nondegenerate, its inverse is given by:

$$B_0^{-1} = \begin{pmatrix} 1 & 0 \\ -M & 1 \end{pmatrix}.$$

In matrix form, the linear system approximating the optimality conditions has the form:

$$\begin{pmatrix} \tilde{k}_{t+1} - \tilde{k}_{ss} \\ \tilde{c}_{t+1} - \tilde{c}_{ss} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} & -\frac{1+\tau^c}{(1+n)(1+\gamma)} \\ -\frac{M}{\beta} & 1 + \frac{M(1+\tau^c)}{(1+n)(1+\gamma)} \end{pmatrix} \begin{pmatrix} \tilde{k}_t - \tilde{k}_{ss} \\ \tilde{c}_t - \tilde{c}_{ss} \end{pmatrix} = D \begin{pmatrix} \tilde{k}_t - \tilde{k}_{ss} \\ \tilde{c}_t - \tilde{c}_{ss} \end{pmatrix}, \tag{25}$$

where the coefficient matrix $D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$ has the characteristic equation:

$$\mu^2 - (d_{11} + d_{22})\mu + (d_{11}d_{22} - d_{12}d_{21}) = 0, \tag{26}$$

with roots:

$$\mu_{1,2} = \frac{(d_{11}+d_{22}) \pm \sqrt{(d_{11}+d_{22})^2 - 4(d_{11}d_{22} - d_{12}d_{21})}}{2}. \tag{27}$$

To analyze the dynamic properties of the system, we apply the spectral decomposition of the matrix D:

$$D = \Gamma\Lambda\Gamma^{-1},$$

where Λ is the diagonal matrix of eigenvalues and Γ is the matrix of right eigenvectors (for a detailed description, see the mathematical appendix [5, p. 201]).

Denote the matrix of right eigenvectors as $\Gamma = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$ and its inverse as $\Gamma^{-1} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$. Then the dynamics of the system can be written as:

$$\begin{pmatrix} \tilde{k}_t - \tilde{k}_{ss} \\ \tilde{c}_t - \tilde{c}_{ss} \end{pmatrix} = \Gamma\Lambda\Gamma^{-1} \begin{pmatrix} \tilde{k}_{t-1} - \tilde{k}_{ss} \\ \tilde{c}_{t-1} - \tilde{c}_{ss} \end{pmatrix}. \tag{28}$$

Equivalently, the dynamics can be written as:

$$\begin{pmatrix} \tilde{k}_t - \tilde{k}_{ss} \\ \tilde{c}_t - \tilde{c}_{ss} \end{pmatrix} = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \begin{pmatrix} \mu_1^t & 0 \\ 0 & \mu_2^t \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} \tilde{k}_0 - \tilde{k}_{ss} \\ \tilde{c}_0 - \tilde{c}_{ss} \end{pmatrix}, \tag{29}$$

where $x_1 = 1$, $x_2 = \frac{\mu_1 - d_{11}}{d_{21}}$, $y_1 = 1$, $y_2 = \frac{\mu_2 - d_{11}}{d_{21}}$,

$$u_1 = \frac{\mu_2 - d_{22}}{\mu_2 - \mu_1}, \quad u_2 = \frac{-d_{12}}{\mu_2 - \mu_1}, \quad v_1 = \frac{\mu_1 - d_{22}}{\mu_1 - \mu_2}, \quad v_2 = \frac{-d_{12}}{\mu_1 - \mu_2}. \tag{30}$$

Fixing the initial values $\tilde{k}_0$ and $\tilde{c}_0$ and applying (28) sequentially, we obtain the entire time path of the state and control variables:

$$\begin{pmatrix} \tilde{k}_t - \tilde{k}_{ss} \\ \tilde{c}_t - \tilde{c}_{ss} \end{pmatrix} = \Gamma\Lambda^t\Gamma^{-1} \begin{pmatrix} \tilde{k}_0 - \tilde{k}_{ss} \\ \tilde{c}_0 - \tilde{c}_{ss} \end{pmatrix}. \tag{31}$$

A saddle-path model is a dynamic system in which the equilibrium has mixed stability: along some directions, deviations dampen, and along others, they diverge. For the Ramsey–Cass–Koopmans model, this means that there is a single stable trajectory that leads the economy to a steady state. The dimension of the stable manifold is equal to the number of stable eigenvalues. The eigenvalues must satisfy saddle-path stability conditions: $|\mu_1| > 1/\beta$ (unstable mode), $|\mu_2| < 1$ (stable mode). The shift of the «instability» threshold to $1/\beta$ is due to the transversality condition $\lim_{t\to\infty} \beta^t \lambda_t \tilde{k}_{t+1} = 0$: since $|\beta\mu_1| > 1$, the contribution does not decay, so the coefficient on this mode should be zero.

Multiplying the matrices in (30), we obtain the following expressions:

$$\tilde{k}_t - \tilde{k}_{ss} = x_1\mu_1^t\left[u_1(\tilde{k}_0 - \tilde{k}_{ss}) + v_1(\tilde{c}_0 - \tilde{c}_{ss})\right] + y_1\mu_2^t\left[u_2(\tilde{k}_0 - \tilde{k}_{ss}) + v_2(\tilde{c}_0 - \tilde{c}_{ss})\right], \tag{32}$$

$$\tilde{c}_t - \tilde{c}_{ss} = x_2\mu_1^t\left[u_1(\tilde{k}_0 - \tilde{k}_{ss}) + v_1(\tilde{c}_0 - \tilde{c}_{ss})\right] + y_2\mu_2^t\left[u_2(\tilde{k}_0 - \tilde{k}_{ss}) + v_2(\tilde{c}_0 - \tilde{c}_{ss})\right]. \tag{33}$$

The transversality condition for the capital stock notes that consumption does not converge to zero along the optimal trajectory:

$$\lim_{t\to\infty} \beta^t \frac{1}{1+\tau^c} \tilde{c}_t^{-\sigma} \tilde{k}_{t+1} = 0 \;\; \Rightarrow \;\; \lim_{t\to\infty} \beta^t \tilde{k}_{t+1} = 0. \tag{34}$$

Since $|\beta\mu_1| > 1$ and $|\beta\mu_2| < \beta$, the transversality condition is fulfilled only when the coefficient at the unstable eigenvalue μ_1 in the equation for $\tilde{k}_t - \tilde{k}_{ss}$ is equal to zero [5, p. 189]. And since $x_1 = 1$ and $\mu_1^t \neq 0$, the expression in brackets in equations (32) – (33) must be equal to zero:

$$u_1(\tilde{k}_0 - \tilde{k}_{ss}) + v_1(\tilde{c}_0 - \tilde{c}_{ss}) = 0. \tag{35}$$

Thus, the stability condition requires that initial consumption be chosen such that:

$$\tilde{c}_0 - \tilde{c}_{ss} = -\frac{u_1}{v_1}(\tilde{k}_0 - \tilde{k}_{ss}) = \frac{\mu_2 - d_{11}}{d_{12}}(\tilde{k}_0 - \tilde{k}_{ss}). \tag{36}$$

The subsequent dynamics of the system are described as:

$$\tilde{k}_t - \tilde{k}_{ss} = y_1 \mu_2^t [u_2(\tilde{k}_0 - \tilde{k}_{ss}) + v_2(\tilde{c}_0 - \tilde{c}_{ss})], \tag{37}$$

$$\tilde{c}_t - \tilde{c}_{ss} = y_2 \mu_2^t [u_2(\tilde{k}_0 - \tilde{k}_{ss}) + v_2(\tilde{c}_0 - \tilde{c}_{ss})] \tag{38}$$

It follows that:

$$\tilde{c}_t - \tilde{c}_{ss} = \frac{y_2}{y_1}(\tilde{k}_t - \tilde{k}_{ss}) = \frac{\mu_2 - d_{11}}{d_{12}}(\tilde{k}_t - \tilde{k}_{ss}). \tag{39}$$

Equation (39) is the stability condition that requires that the proportional relationship between deviations in capital and consumption be maintained at each point in time, starting at $t = 0$.

Thus, the dynamics of the system can be represented in compact form [5, p. 190]:

$$\tilde{k}_t - \tilde{k}_{ss} = \mu_2^t \left[y_1 \left(u_2(\tilde{k}_0 - \tilde{k}_{ss}) + v_2(\tilde{c}_0 - \tilde{c}_{ss}) \right) \right] = \mu_2^t (\tilde{k}_0 - \tilde{k}_{ss}), \tag{40}$$

$$\tilde{c}_t - \tilde{c}_{ss} = \frac{\mu_2 - d_{11}}{d_{12}} \mu_2^t (\tilde{k}_0 - \tilde{k}_{ss}) = \mu_2^t (\tilde{c}_0 - \tilde{c}_{ss}). \tag{41}$$

Numerical solution of the model. To quantitatively analyze the impact of technological progress within the proposed model, we consider a numerical solution using specific values of structural parameters. At each time step, the level of consumption $\tilde{c}_t$ (the control variable) is chosen in accordance with the stability condition (39) and the aggregate resource constraint (3) – under these conditions, as shown above, the solution converges to the steady state.

For the numerical experiment, we set the reference values of the structural parameters for the European Union (EU) and the United States (US), and justify these choices (Table 1).

Table 1. Typical values of structural parameters for the EU and US economies.

	β	δ	φ	α	n	σ	γ	τ^y	τ^c
EU	0,9920	0,1000	1	0,36	0,003	1,613	0,0120	0,108	0,183
USA	0,9984	0,0963	1	0,30	0,009	1,380	0,0177	0,065	0,077

Source: compiled from the cited sources [7–14].

There are several main approaches to estimating the discount factor β: through the long-term real interest rate r; through panel estimates based on household expenditure and consumption data; through surveys or experimental studies that reflect individuals' time preferences. Our analysis uses quarterly values of β based on DSGE model estimates: $\beta = 0.9996$ for the US economy [7, p. 22] and $\beta = 0.998$ for the EU [8, p. 87]. Such values of the discount factor are typical for models that reproduce low long-term real rates and moderate growth rates.

Drawing on the same sources, we obtain the following values for the other structural parameters:

- depreciation rate (δ): US – 9.63% per year (2.5% per quarter) [7, p. 22]; EU – 10% per year [8, p. 41];
- share of capital in the production function (α): US – 0.3 [7, p. 23]; EU – 0.36 [8, p. 41];
- annual growth rate of labor efficiency (γ): US – 1.77% [7, p. 22]; EU – 1.2% [8, p. 38];
- population growth rate (n): EU – average of 0.3% per year [8, p. 41]; US – 0.9% per year in 1974–2023, with a forecast of about 0.6% in 2024–2034 [9].

In our specification $Y_t = \varphi K_t^{\alpha}(A_t L_t)^{1-\alpha}$, the parameter φ defines the level (scale) of total factor productivity (TFP); it can be normalized to one without loss of generality. We draw conclusions about technological progress primarily by analyzing the Harrod-neutral (labor-augmenting) progress rate γ.

The CRRA parameter σ was estimated using Bayesian methods via the Metropolis–Hastings algorithm and equals 1.380 for the US [10, p. 593] and 1.613 for the EU [11, p. 40].

Regarding the fiscal parameters, the consumption tax rate τ^c for the United States is 0.077 and for the EU is 0.183 [12, p. 169]. For an ad valorem output tax rate τ^y for the EU, we use Eurostat's measure of taxes on production and imports net of subsidies – 10.8% in 2024 [13]. For the United States, we use the share of production and import taxes in gross national income of 6.9% [14], subtract the share of subsidies of 0.4% [15], and obtain an approximate net value of 6.5% (2023). Note that this is an approximation: the data tax base differs from the models, but this mapping is convenient in macro models.

To study the effects of technological progress, we model the dynamics over an 80-year horizon using the parameters from Table 1, except for γ, which is set to zero in the first 10 years (no technological progress). We calculate the steady-state levels of per-effective-worker effective capital endowment $\tilde{k}_{ss}$ and consumption $\tilde{c}_{ss}$; the initial value is $\tilde{k}_0 = \tilde{k}_{ss}$. In per-effective-worker terms, in steady-state equilibrium, the values of output, consumption, and capital do not change. For each subsequent period, we calculate $\tilde{k}_{t+1}$ from the resource constraint (3), and determine $\tilde{c}_t$ from the stable-path equation (39), which guarantees convergence to the steady state [5, p. 191]; after changing γ, we recalculate the new $\tilde{k}_{ss}(\gamma)$, $\tilde{c}_{ss}(\gamma)$ and the new slope of the stable path. Note that the Harrod-neutral representation of technological progress and working in per-effective-worker units yield stable long-run steady states for the listed variables, while the levels (in natural units) grow at rate $(1+n)(1+\gamma)$.

The purpose of the experiment is to quantify the consequences of introducing technological progress for key long-run macroeconomic indicators in the US and the EU – consumption and output per capita, capital-labor productivity, and, crucially in the neoclassical model, aggregate discounted utility (welfare).

Note that, to measure welfare, it is more appropriate to move from per-effective-worker variables back to per-worker values. The utilitarian social welfare function describes the welfare of real people, so the natural argument of utility is consumption per worker $c_t \equiv C_t/N_t$, rather than per effective worker, $\tilde{c}_t \equiv C_t/(A_t N_t)$. Accordingly, for empirical analysis and inference, we re-express the variables k_t, c_t, y_t, using the relations $c_t = A_t\tilde{c}_t$, $k_t = A_t\tilde{k}_t$, $y_t = A_t\tilde{y}_t$.

Since a change in any structural parameter moves the economy out of its steady state and it transitions to a new steady state. For parameters with a pronounced transitional dynamics (γ, τ^y), we present time-path of deviations (Table 2), and for parameters with a predominantly long-run effect $(\delta, \alpha, n, \tau^c, \beta)$ – we report steady-state changes in levels (Table 3). The tables report percentage changes in levels relative to the baseline trajectory.

Table 2 shows that technological progress is one of the key parameters that determine the levels of macroeconomic indicators in the EU and the US. In particular, a 10% increase in the rate of Harrod-neutral technological progress (relative to the baseline) increases consumption per worker by about 1–1.5% in 10 years, by 3–5% in 30 years, and by 8–12% in 70 years; similar conclusions apply to capital stock and output per capita. The impact of γ is nonlinear (growing over time), which reflects the specifics of the Ramsey–Cass–Koopmans model with Harrod-neutral progress: the growth factor A_t amplifies the deviation of levels over time. This is clearly seen in Figures 1–2, which show that the increase in capital per worker due to an increase in γ is a convex, increasing function of t, with convexity rising with larger γ.

Table 2. Sensitivity of $\Delta c, \Delta k, \Delta y$ per worker in response to a 10% increase in each of the parameters γ and τ^y at different time horizons

Countries	Parameter change	Time horizon, years	γ(+10%)	τ^y(+10%)
EU	Δc	10	+0.98%	–1.57%
		30	+3.18%	–1.84%
		70	+8.15%	–1.88%
	Δk	10	+0.34%	–1.19%
		30	+2.24%	–1.79%
		70	+7.13%	–1.88%
	Δy	10	+0.96%	–0.43%
		30	+3.20%	–0.65%
		70	+8.18%	–0.68%
USA	Δc	10	+1.54%	–0.88%
		30	+4.90%	–0.98%
		70	+12.44%	–0.99%
	Δk	10	+0.42%	–0.73%
		30	+3.45%	–0.97%
		70	+10.85%	–0.99%
	Δy	10	+1.47%	–0.22%
		30	+4.90%	–0.29%
		70	+12.44%	–0.30%

Source: author's calculations.

Table 3. Percentage changes in $\Delta c, \Delta k, \Delta y$ per worker in response to a 10% increase in parameters (δ, α, n, τ^c) and a 10% decrease in β

Countries	Parameter change	δ(+10%)	α(+10%)	n(+10%)	τ^c(+10%)	β(-10%)
EU	Δc	–4.55%	+10.49%	–0.15%	–1.52%	–13.95%
	Δk	–11.48%	+28.02%	–0.39%	–0.00%	–63.99%
	Δy	–4.29%	+16.38%	–0.14%	–0.00%	–30.77%
USA	Δc	–3.17%	+5.74%	–0.32%	–0.71%	–9.10%
	Δk	–10.07%	+21.42%	–1.04%	–0.00%	–60.49%
	Δy	–3.14%	+10.38%	–0.31%	–0.00%	–24.32%

Source: author's calculations.

The United States realizes larger level gains from an increase in the rate of technological progress than the EU. This is due to the chosen structural parameters: the US has a higher underlying rate of technological progress ($\gamma = 0{,}0177$ vs. $0{,}0120$), so the cumulative effect of

A_t is stronger; a lower output tax rate ($\tau^y = 0{,}065$ vs. $0{,}108$); higher patience ($\beta = 0.9984$ vs. 0.9920), lower depreciation ($\delta = 0{,}9963$ vs. $0{,}1000$); and a higher intertemporal elasticity of substitution (because $\sigma_{US} = 1{,}380 < \sigma_{EU} = 1{,}613$). Taken together, these parameters amplify the response of saving and capital to an increase in γ, so that long-term gains in y, c, k and welfare (as will be demonstrated below) are higher in the United States – even though the capital share α is higher in the EU (0.36 vs. 0.30).

One of the advantages of the neoclassical Ramsey–Cass–Koopmans model is the ability to assess the impact of changes in structural parameters on welfare in terms of aggregate discounted utility. At the same time, «utility» is an abstract indicator. To make the interpretation meaningful, we use the Lucas (consumption-equivalent welfare) approach [16].

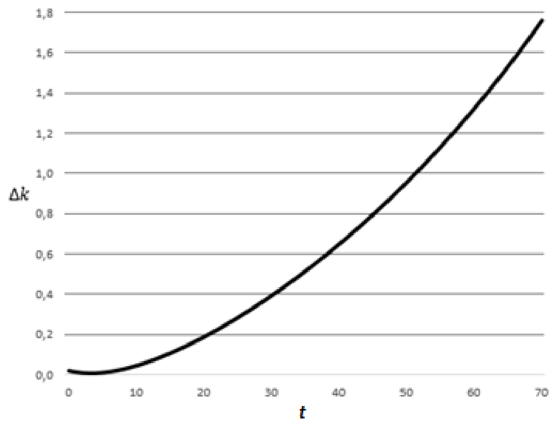

Figure 1. Increase in capital per worker Δk (% of the baseline scenario) as a result of a twofold increase in the rate of technological progress (based on typical structural parameters for the United States; new value $\gamma = 0.0354$)

Source: author’s illustration.

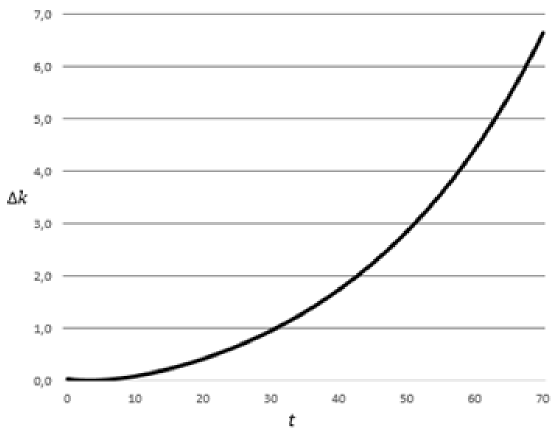

Figure 2. Increase in capital per worker Δk (% relative to the baseline scenario) as a result of a threefold increase in the rate of technological progress (based on typical structural parameters for the United States, new value $\gamma = 0.0531$)

Source: author’s illustration.

The idea is to find a constant multiplicative scaling of consumption in the baseline scenario that makes the agent indifferent between the two trajectories. This approach is useful for comparing alternative policies (e.g., changes in tax rates). We adapt Lucas's method to estimate welfare gains from an increase in the rate of technological progress, and denote this consumption-equivalent factor by λ.

The intuition is as follows [16, p. 1]:

$$U\big((1+\lambda)c^A\big) = U(c^B), \tag{42}$$

that is, λ shows by how much consumption should be permanently multiplied in the baseline scenario c^A for the representative agent to be indifferent between it and the alternative path c^B.

For CRRA preferences (for $\sigma \neq 1$), the coefficient λ is calculated as:

$$\lambda = \left(\frac{\sum_{t=0}^{T}\beta^t\left(c_t^B\right)^{1-\sigma}}{\sum_{t=0}^{T}\beta^t\left(c_t^A\right)^{1-\sigma}}\right)^{\frac{1}{1-\sigma}} - 1, \tag{43}$$

where c_t^A is the baseline trajectory of consumption per worker; c_t^B is the trajectory of consumption after the increase in γ. If $\lambda > 0$, the policy B increases welfare; in the B scenario, consumption can be reduced by a fraction of $\lambda/(1+\lambda)$ each year and the agent remains indifferent to the baseline scenario.

Importantly, we do not compute welfare over an infinite horizon, but only consider the first 70 years; because β is close to one, the unaccounted-for residual utility can be significant. For the parameters from Table 1, we obtain:

- US: $\lambda_{10}^{US} = 0.0092$, $\lambda_{30}^{US} = 0.0232$, $\lambda_{70}^{US} = 0.0528$;
- EU: $\lambda_{10}^{EU} = 0.0057$, $\lambda_{30}^{EU} = 0.0149$, $\lambda_{70}^{EU} = 0.0349$.

Let us interpret the results using $\lambda_{70}^{USA} = 0.0528$: a 10% increase in the rate of technological progress is equivalent to a permanent 5.28% increase in consumption in the baseline scenario over the first 70 years. This quantitatively confirms the positive impact of exogenous technological progress on welfare in the Ramsey–Cass–Koopmans model.

Conclusions. The study analyzes the role of exogenous technology in the neoclassical Ramsey–Cass–Koopmans model with Harrod-neutral technological progress. On the basis of reasonable typical values of structural parameters for the EU and US economies, a numerical experiment is conducted to assess the consequences of a relative increase in the rate of technological progress at horizons of 10, 30, and 70 years.

Changes in the levels of consumption and output per capita, as well as capital intensity, are examined. The sensitivity results indicate that technological progress is one of the key determinants of long-term growth outcomes in modern Western economies. In particular, 70 years after a 10% increase in γ, we observe an increase in consumption per worker of about 8–12%, capital intensity of 7–11%, and output per worker of 8–12% (compared to the baseline scenario).

In contrast to changes in the depreciation rate δ, the capital share α, the population growth rate n and the discount factor β, the impact of technological progress is nonlinear (increasing over time): the growth of capital stock as a function of the rate of technological progress is convex, and the convexity increases with γ.

Given that the Ramsey–Cass–Koopmans model optimizes the growth trajectory according to the utilitarian social welfare function, we estimate the welfare implications of increasing γ using the Lucasian approach. The idea is to measure the constant percentage change in consumption in the baseline scenario that makes the representative agent indifferent between the baseline and alternative trajectories. For our initial parameters, a 10% increase in the rate of technological progress is equivalent to a permanent increase in baseline consumption of 5.28% in the US and 3.49% in the EU over the first 70 years to remain on the same indifference curve. This quantitatively confirms the positive impact of exogenous technological progress on welfare.

Further research should analyze the impact of modern digital technologies (in particular, artificial intelligence) on growth models, both with exogenous technological progress and with endogenous innovation capturing microeconomic agents' incentives. In this context, the Ramsey–Cass–Koopmans model serves as a workhorse baseline for further modifications and extensions.

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