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STRESS–STRAIN STATE AND ELASTIC PROPERTIES OF COMPOSITES WITH VARIABLE REINFORCEMENT STRUCTURE

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Abstract. *A numerical and analytical investigation of the influence of spatial variation in reinforcement density on the stress–strain state (SSS) and effective elastic characteristics of unidirectional fibrous composites is presented. The importance of this work is due to the fact that in modern engineering applications (aerospace, automotive, and shipbuilding), the reinforcement density often varies along one or several coordinates for components of complex shape or variable thickness. This affects the stress–strain state and the accuracy of analytical estimates based on classical averaged models. The objective of the paper is to formulate and analyze relationships for the effective characteristics of an orthotropic layer with a variable reinforcement coefficient. Based on the rule of mixtures (Voigt and Reuss estimates), refined relationships for the components of the stiffness matrix of the orthotropic layer were derived. These relationships take into account the variable reinforcement coefficient along the height of the cross-section, and a constant coefficient along the width. Verification and comparison of the obtained relationships with simplified averaged formulas and 3D finite element method (FEM) calculations performed in Ansys Workbench were carried out. For the FEM modeling, a representative volume of trapezoidal shape (height 200 mm, bases 30 mm and 90 mm, thickness 30 mm) was considered in the tension problem (one end fixed, the load of 40 kN applied at the other end). Materials: carbon fiber and matrix. Three models were constructed: (1) with geometrically separated components, (2) equivalent orthotropic with refined stiffness matrix components, (3) equivalent orthotropic with simplified formulas. It is shown that taking into account the variability of reinforcement density in two directions reduces the error in determining displacements and moduli. The refined formulas demonstrate a significant reduction in errors in the «quasi-homogeneous» interior region of the sample: the average relative displacement error is about 2% for the refined formulas compared to 5% for the simplified ones. At the ends of the sample, due to edge effects, the errors reach up to 30%. The limits for the correct application of averaged relationships in tension problems were determined (for the given sample, the interval is 75–190 mm), and directions for extrapolation to bending were outlined. This requires a separate sensitivity analysis with respect to shear characteristics.*

Key words: *fibrous composites, anisotropy, rule of mixtures, stiffness matrix, modulus of elasticity, finite element method.*

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1. INTRODUCTION

Modern engineering applications of fibrous composites (aerospace, automotive, shipbuilding) rely on the ability to shape desirable mechanical properties by choosing the reinforcement architecture and the element geometry. The key parameter is the fiber volume fraction, or reinforcement coefficient, which determines the effective moduli, Poisson's and shear ratios. In practice, particularly for components of complex shape or variable thickness, the reinforcement density varies along one or more coordinates, which affects the stress-strain state and the accuracy of analytical estimates based on classical averaged models.

The objectives of this paper are: (i) to formulate and analyze relationships for the effective characteristics of the orthotropic layer with variable reinforcement coefficient along characteristic coordinate; (ii) to verify these relationships using 3D FEM model of representative volume with variable cross-section; (iii) to determine the limits of correctness of the averaged formulas in the tension problem.

1.1. Literature review and problem statement

The theory and engineering models of the effective elastic properties of unidirectional composites traditionally rely on both macro- and microscopic approaches, ranging from classical textbooks and monographs to applied developments for structures with variable stiffness [1–7, 9–15]. Macroscopic approaches (the rule of mixtures and its modifications) provide simple closed formulae for $E_1, E_2, G_{12}, \nu_{12}$ assuming homogeneous fiber distribution. Microscopic approaches and numerical models make it possible to take into account geometry, fiber–matrix contact, and local effects; however, they are sensitive to discretization and connection conditions.

In products with variable geometry (conical and spherical shells, panels of varying thickness), the distance between fibers changes systematically, and consequently, the local fraction of the reinforcing phase varies. This poses the problem of correct transitioning from local constituents to the effective characteristics of the layer with reinforcement parameter ψ as a function of coordinate, and of estimating the errors associated with the application of simplified averaged models.

2. METHODOLOGY AND MATHEMATICAL MODEL

2.1. Geometry, materials, and boundary conditions

The representative volume (sample) of trapezoidal shape: top base 30 mm, bottom base 90 mm, height 200 mm, thickness 30 mm was considered. Reinforcement consists of carbon fibers with square cross-section, 10 mm on each side.. Mechanical properties: for fiber, modulus of elasticity is $E_1 = 300\,000$ MPa, Poisson's ratio $\nu_1 = 0.28$v; for the matrix, $E_2 = 4\,000$ MPa, $\nu_2 = 0.22$. FEM mesh is tetrahedra with characteristic size of 2 mm. Boundary conditions: one end is fixed; uniformly distributed load of 40 kN is applied at the free end. Along the fiber axis (height), the effective fiber fraction $\psi_3(x_1) \in [0.111, 0.333]$ varies; in the width direction, constant coefficient $\psi = 0.333$ is assumed.

2.2. Averaged relations and reinforcement variability

For unidirectional layer with regular arrangement of fibers, ideal contact, and small deformations, Hooke's law in the plane stress state is expressed as

$$\{\sigma\} = [C]\{\varepsilon\},$$

$$\{\sigma\} = (\sigma_{11}, \sigma_{22}, \tau_{12})^{\mathsf{T}},$$

$$\{\varepsilon\} = (\varepsilon_{11}, \varepsilon_{22}, \gamma_{12})^{\mathsf{T}},$$

where the stiffness matrix of the orthotropic layer has the following form

$$[C] = (c_{11}\; c_{12}\; 0\; c_{12}\; c_{22}\; 0\; 0\; 0\; c_{66}).$$

The generalized rule of mixtures linearly interpolates the properties of the composite according to the fiber volume fraction V. For the direction along the fibers (Voigt estimate)

$$E_1(x_1) = E_F\, V(x_1) \text{i} E_M\, [1 - V(x_1)],$$

for the direction across the fibers (Reuss estimate)

$$\frac{1}{E_2(x_1)} = \frac{V(x_1)}{E_F} + \frac{1 - V(x_1)}{E_M},$$

and for the shear modulus in the plane of layer $G_{12}(x_1)$ the shear moduli for the individual components (fiber and matrix) are determined the first. They are calculated using the general formula for isotropic material, which relates the shear modulus to the elastic (Young's) modulus and Poisson's ratio:

$$G = \frac{E}{2(1+\nu)}$$

Thus, using the properties of fiber $(E_{,}\nu_F)$ and matrix $(E_{,}\nu_M)$, we obtain:

$$G_F = \frac{E_F}{2(1+\nu_F)} \text{ та } G_M = \frac{E_M}{2(1+\nu_M)}$$

The effective shear modulus of the composite $G_{12}(x_1)$, according to the inverse rule of mixtures (Reuss estimate), is defined as:

$$\frac{1}{G_{12}(x_1)} = \frac{V(x_1)}{G_F} + \frac{1-V(x_1)}{G_M}$$

The effective Poisson's ratios can be estimated as:

$$\nu_{12}(x_1) = \nu_F\, V(x_1) + \nu_M\,[1 - V(x_1)],$$

$$\nu_{21}(x_1) = \nu_{12}(x_1)\,\frac{E_2(x_1)}{E_1(x_1)}.$$

In the presence of the variable reinforcement fraction along x_1 coordinate, a separate consideration of reinforcement is introduced with two: $\psi_3(x_1)$ – the variable reinforcement coefficient along the height of the cross-section, and a constant coefficient ψ along the width of the section. The equivalent parameter for averaging over the cross-section is: $V(x_1) = \psi\,\psi_3(x_1)$.

Refined expressions for $c_{11}, c_{22}, c_{12}, c_{66}$ as functions $E_1, E_2, \nu_1, \nu_2, V(x_1)$ are obtained by substituting the given relationships into the corresponding equations for the strain characteristics, followed by the calculation of the stiffness coefficients [8]. For further comparison, simplified formulas are also used to calculate the modulus of elasticity in the characteristic directions within the plane of the cross-section:

$$E_4^{(smpl)}(x_1) = \frac{E_1\,E_2}{\psi\,E_2 + [1-\psi]\,E_1}, \tag{1}$$

$$E_3^{(smpl)}(x_1) = E_1\,\psi + E_2\,[1-\psi], \tag{2}$$

which do not take into account the dependence $\psi_3(x_1)$, assuming that ψ remains constant along both axes in the plane of the cross-section.

Three calculation models were constructed:

1. The sample with geometrically separated composite components (fiber + matrix) and variable cross-section;

2. The equivalent orthotropic model with constant cross-section and refined $[C(x_1)]$, calculated discretely at five sections based on $\psi_3(x_1)$;

3. The equivalent orthotropic model with constant cross-section, using the simplified averaged formulas (1), (2).

For each setup, analysis of the displacement distribution along the sample axis was carried out, and the relative errors of the results obtained from the simplified models 2 and 3 were evaluated with respect to the sample model with geometrically separated composite components (model 1).

3. RESULTS

3.1. Effective moduli according to different approaches

Comparative estimates of the moduli (MPa) were obtained for the characteristic sections of the sample. The table shows the example of the correspondence from five control coordinates along the axis (40–200 mm) for two approaches: simplified and refined.

Table 1

Calculated elastic moduls results using different methods for determining elastic properties

Coordinate along the rod axis, mm	E3, (2)–(3)	E3, (1)	E4, (2)-(3)	E4, (1)
40	29517	27596	5228	4486
80	23474	22349	4894	4346
120	19745	19011	4703	4277
160	17214	16701	4579	4237
200	15385	15007	4493	4211

3.2. Displacement fields and errors

Figure 1 shows The calculation 3D model with the overall displacement is shown in Fig. 1. The displacement graphs along the central axis (Fig. 2) demonstrate systematic difference between the simplified and refined approaches compared to the designed model:

- At the ends of the rod, the relative errors reach approximately 30%.
- Inside the rod, the error does not exceed 20% for the refined formulas and 25% for the simplified ones.
- The area with the error below 20% covers the part of the sample in the coordinate range along the rod axis of 75–190 mm, corresponding to values $0{,}16 < \psi_3(x_1) < 0{,}32$.
- The average relative displacement error in this area is approximately 2% for the refined formulas and 5% for the simplified ones.

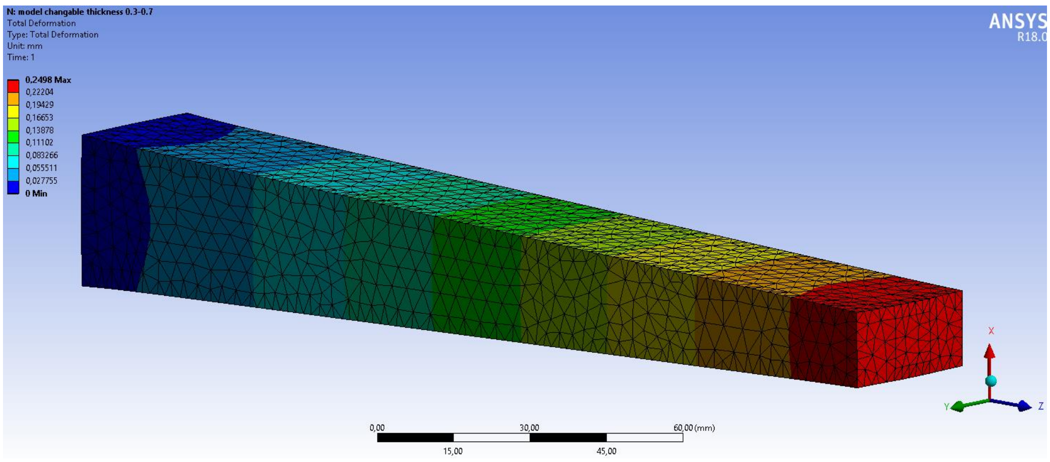

Figure 1. Displacement along the axis for the sample with variable thickness

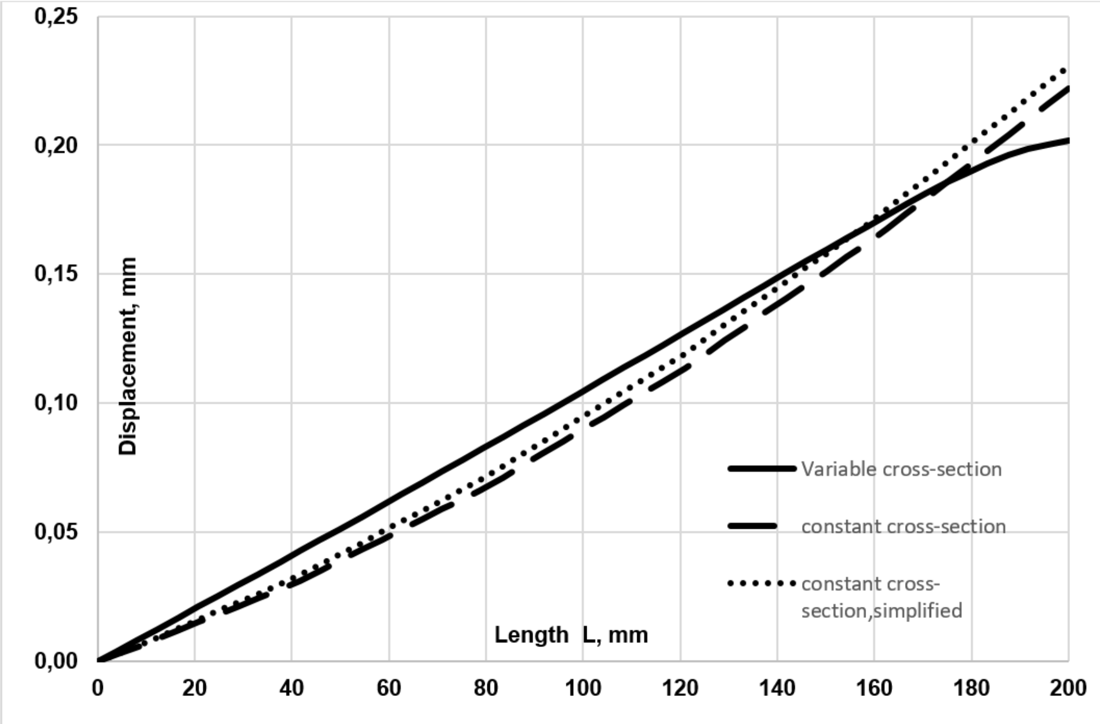

Figure 2. Calculation results

3.3. Analysis of the results

The variation in reinforcement density causes fluctuations in local stiffness and, accordingly, an uneven distribution of deformations. The simplified models with constant ψ overestimate stiffness in areas with reduced fiber fraction and underestimate it in areas with increased fraction, which accumulates into global displacement deviations. The refined formulas that take into account $\psi_3(x_1)$ combination with constant ψ, show a significant reduction in errors in the «quasi-homogeneous» internal region of the sample.

The largest discrepancies (up to 30 %) are observed at the ends of the sample.. The central part, where $\psi_3(x_1)$. has intermediate values ($\approx 0.16 \ldots 0.32$), is characterized by smaller deviations – no more than 20 % for the refined formulas and around 25 % for the simplified ones.

In practice, this means that for elements of variable thickness or shells of variable thickness, spatially varying effective properties should be used when constructing orthotropic layers, and the discretization scale should be considered relative to the gradient $\psi_3(x_1)$.

Extending the conclusions to bending requires a separate analysis due to the change in the ratio of normal to shear strains, as well as the sensitivity to $G_{12}(x_1)$ and interlayer interactions. As it is expected, the difference between the approaches will be greater in problems where shear stresses and torsional components dominate.

4. CONCLUSIONS

- The consistent scheme for determining the effective elastic properties of unidirectional layer with variable reinforcement coefficient along the characteristic coordinate, based on the rule of mixtures and the relationships between local stress–strain states has been formulated. This approachmakes it possible to take into account structural inhomogeneity and ensures the correct determination of macroscopic elastic moduli and Poisson's ratios for subsequent use in the analysis of multilayer composite elements.

- In Ansys Workbench, for the sample with variable cross-section, the refined relations provide the average displacement error of about 2 % in the internal area, compared to 5 % for the simplified formulas; at the ends, errors increase to approximately 30 % due to edge effects.
- Consideration of variations in reinforcement density in two directions (height and width) systematically reduces discrepancies and improves the predictability of the stress-strain state.
- The operating range of validity for the averaged formulas in the tensile problem (lengths of 75–190 mm for the considered sample) has been determined. For bending problems, additional investigations, focusing on the role of $G_{12}(x_1)$, fiber-matrix contact interactions, and interlayer stress transfer are required.

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НАПРУЖЕНО-ДЕФОРМОВАНИЙ СТАН І ПРУЖНІ ВЛАСТИВОСТІ КОМПОЗИТІВ ЗІ ЗМІННОЮ СТРУКТУРОЮ АРМУВАННЯ

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Резюме. *Подано чисельне та аналітичне дослідження впливу просторової змінності щільності армування на напружено-деформований стан (НДС) і ефективні пружні характеристики односпрямованих волокнистих композитів. Актуальність роботи зумовлена тим, що в сучасних інженерних застосуваннях (авіа-, авто-, суднобудування) для елементів складної форми чи змінної товщини щільність армування часто змінюється вздовж однієї або кількох координат. Це впливає на НДС і точність аналітичних оцінок за класичними усередненими моделями. Метою роботи було формулювання та аналіз співвідношень для ефективних характеристик ортотропного шару за змінного коефіцієнта армування. На основі правила сумішей (оцінок Фойгта та Ройса) сформовано уточнені співвідношення для компонент матриці жорсткості ортотропного шару. Ці співвідношення враховують змінний коефіцієнт армування по висоті та сталий по ширині поперечного перерізу. Проведено верифікацію та порівняння отриманих співвідношень зі спрощеними усередненими формулами та 3D-розрахунком методом скінченних елементів (МСЕ) в Ansys Workbench. Для МСЕ-моделювання розглянуто представницький об'єм трапецеподібної форми (висота 200 мм, основи 30 і 90 мм, товщина 30 мм) у задачі розтягу (затиснення одного кінця, навантаження 40 кН на іншому). Матеріали: вуглецеве волокно та матриця. Побудовано три моделі: (1) з геометрично розділеними компонентами, (2) еквівалентна ортотропна з уточненими компонентами матриці жорсткості, (3) еквівалентна ортотропна зі спрощеними формулами. Показано, що врахування змінності щільності армування у двох напрямках зменшує похибку визначення переміщень і модулів. Уточнені формули демонструють суттєве зменшення похибок у «квазіоднорідній» внутрішній області зразка: усереднена відносна похибка переміщень тут становить близько 2% для уточнених формул проти 5% для спрощених. На кінцях зразка, через крайові ефекти, похибки сягають 30%. Визначено межі коректного застосування усереднених співвідношень у задачах розтягу (для даного зразка – інтервал 75–190 мм) та окреслено напрями екстраполяції на згин, що вимагає окремого аналізу чутливості до характеристик зсуву.*

Ключові слова: *волокнисті композити, анізотропія, правило сумішей, матриця жорсткості, модуль пружності, метод скінченних елементів.*

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