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THE THEORY OF SPLINE INTERFLATATION OF FUNCTIONS r VARIABLES r ≥ 2

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Abstract. *The theory of approximation of multidimensional functions of r variables r ≥ 2 using spline-interflatation operators is developed in this paper. A new method for constructing such operators, which is based on the approach of decomposing the multidimensional approximation problem into a sequence of one-dimensional problems, each solved using spline interpolation. is proposed in this paper. This makes it possible to investigate the interflatation properties of the constructed operators, as well as to analyze their effectiveness in approximating functions with several variables. The distinctive feature of the proposed method is the explicit representation of spline-interflatation operators in terms of one-dimensional spline interpolation operators, which are applied separately to each variable of the approximated function. This provides convenience in investigating the properties of operators and enables more in-depth analysis of their behavior. The expression for the approximation remainder of functions using these operators, in particular, in terms of the remainders that arise from applying one-dimensional spline interpolation operators is investigated in this paper. Special attention is paid to the analysis of approximation remainders of multidimensional functions and to proving that the approximation remainder calculated by means of the proposed interflatation operators is equal to the operator product of the approximation remaindes, defined separately for each variable. This means that total remainder can be considered as a combination of remainders obtained through one-dimensional operators, which significantly simplifies the analysis and makes it possible to investigate the approximation accuracy more thoroughly. Furthermore, a comparative analysis of the obtained results with classical multidimensional interpolation operators is carried out in this paper. This enables us to evaluate the advantages and disadvantages of the proposed method in the context of accuracy and efficiency of approximating functions with several variables. This opens up prospects for further development of the theory of multidimensional approximation and its application in various fields of science and engineerin , where efficient and accurate approximation of multidimensional functions is required.*

Key words: *spline-interflatation, operator, approximation error, interpolation, remainder, differential function, Taylor formula.*

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1. INTRODUCTION

The theory of approximation of differentiable functions r of variables $f(x),\ x=\left(x_1,x_2,...,x_r\right)^T \in D=[0,1]^r,\ r\ge 2$ using specially constructed spline interflatation operators is developed and comprehensively investigated in this paper [1, 2, 3]. The main attention is paid to the investigation of the interflatation properties of the proposed operators, as well as to establishing the relationship between these properties and classical approaches to interpolation and spline interpolation of functions of many variables [4, 5, 6]. The results obtained in this paper are based on modern achievements in the field of approximation theory and spline analysis [5, 6, 7].

The construction of the spline interflatation operators is based on first-degree splines, which are distinguished by their simple structure and low computational cost, making them suitable for practical applications [2, 8]. The main advantage of this approach is that only limited information about the function being approximated is required to construct the corresponding operators – in particular, the value of the function on the edges of the unit cube. This means that multidimensional approximation can be implemented by sequentially applying one-dimensional operators O_k to each variable separately [3, 6, 9].

From the formal point of view, the interflation problem in this context is to construct operators that combine both the properties of interpolation and approximation. On the one hand, these operators preserve the function values at specific points (on the edges of the cube), while on the other hand, they enable us to obtain the global approximation that exhibits good behavior for the class of smooth (differentiable) functions. This highlights the difference between spline interflatation and classical interpolation: instead of requiring exact coincidence with the function at numerous nodes, interflatation operators can use additional flexibility while maintaining selected characteristics of the function [1, 3, 9].

At present, the problem of reconciling classical interpolation and spline interflatation in the context of function approximation on the edges of the unit cube remains insufficiently investigated. The aspect of the similarity between the structural properties of the constructed operators and the properties of the approximation remainders (i.e., errors) arising in the approximation process remains especially insufficiently investigated [4, 10, 11]. This paper makes a significant contribution to filling this gap by proving the theorems that describe explicit expressions for spline interflatation operators in terms of compositions of one-dimensional operators O_k [3, 9]. This makes it possible to reduce the multidimensional problem to a sequence of one-dimensional problems, which significantly simplifies both theoretical analysis and numerical implementation.

Furthermore, it has been proven that the approximation remainder (the difference between the exact function value and the value obtained using the interflatation operator) for the class of differentiable functions can also be represented as the operator product of the one-dimensional approximation remainders [8, 9, 10, 11]. This structure of the remainder term facilitates a detailed analysis of approximation accuracy as well as the development of adaptive algorithms that automatically determine optimal approximation parameters depending on the function properties [3, 12].

In order to assess the effectiveness of the constructed operators, a comparative investigation of the approximation remainders for classical interpolation is carried out. The obtained results made it possible to establish a deep functional connection between the classical spline interpolation problem and the spline interflatation problem. In particular, it has been demonstrated that these problems have a *dual* nature. This means that the explicit expressions for spline interflatation operators, constructed through compositions of one-dimensional operators, turn out to be analogous to the expressions for the remainders of spline interpolation of differentiable functions. And vice versa - the approximation remainder obtained in the case of spline interflatation is expressed through the product of one-dimensional remainders R_k, arising from the application of the corresponding operators to each variable [3, 4, 6].

Special attention should be paid to the fact that classical interpolation operators can also be represented as a product of one-dimensional operators [5, 6]. This property creates opportunities for further generalization of approximation methods, particularly in the direction of developing highly accurate and efficient algorithms for the numerical solution of problems in mathematical physics, multidimensional data processing, machine learning, and other modern fields where precise modeling of complex multivariate dependencies is required [7, 13, 14].

In conclusion, it should be noted that the results obtained in this paper have both theoretical and applied significance. On the one hand, they deepen the understanding of the nature of the relationship between different types of approximation problems, and on the other hand, they create the basis for constructing new approximation methods that combine the advantages of spline interpolation and interflatation [1, 3, 9]. Such methods can be particularly useful in conditions of limited information about the function or when it is necessary to maintain computational efficiency.

2. CONSTRUCTION AND INVESTIGATION OF SPLINE INTERFLATATION OPERATORS FOR FUNCTIONS r OF VARIABLES ($r \geq 2.$)

Let us introduce the notation: $f(x),\ x = \left(x_1, x_2, ..., x_r\right)^T \in D = [0,1]^r,\ r \geq 2$, N is the given natural number, I is the same operator, $h(t)$ is piecewise linear function with properties [1, 3]:

$$h(0) = 1;\ h(t) = 0, |t| \geq 1.$$

For each $k = \overline{1, r}$ let us construct operators O_k of the following form [3]

$$O_k f(x) = \sum_{i_k=0}^{N} f\left(x_1, ..., x_{k-1}, \frac{i_k}{N}, x_{k+1}, ..., x_r\right) \cdot h\left(N \cdot x_k - i_k\right) \quad (1)$$

Let us note that from the definition of the function $h(t)$ it follows that $h\left(N \cdot x_k - i_k\right) = 1, x_k = \frac{i_k}{N};\ h\left(N \cdot x_k - i_k\right) = 0,\ \left|N \cdot x_k - i_k\right| \geq 1.$

Therefore, operator $O_k f(x)$ is the operator of piecewise linear spline interpolation with respect to the variable x_k [4, 5]

$$O_k f(x)\Big|_{x_k = \frac{p_k}{N}} = f(x)\Big|_{x_k = \frac{p_k}{N}},\ p_k = \overline{0, N}.$$

In the case of $r = 2$ these operators will be interlineation operators, and in the case of $r \geq 3$ they will be interflatation operators [1, 3].

It is necessary to construct non-identity operator O with the following properties:

$$O_k f(x)\Big|_{x_k = \frac{p_k}{N}} = f(x)\Big|_{x_k = \frac{p_k}{N}},\ p_k = \overline{0, N}. \quad (2)$$

Let us introduce into consideration operator [3]

$$Of(x) = \left(I - \prod_{k=1}^{r}\left(I - O_k\right)\right) f(x) \quad (3)$$

Theorem 1. Operator O meets the following reguirements: $O_q Of(x) = O_q f(x), q = \overline{1, r}$.
Proof.

$$O_q O f(x) = O_q f(x) - O_q \prod_{k=1}^{r} \left(I - O_k\right) f(x) =$$

$$= O_q f(x) - \left(O_q - O_q O_q\right) \prod_{k=1, k \neq q}^{r} \left(I - O_k\right) f(x) =$$

$$= O_q f(x) - \left(O_q - O_q\right) \prod_{k=1, k \neq q}^{r} \left(I - O_k\right) f(x) = O_q f(x), \ q = \overline{1, r}$$

.

Here we use the fact that the operator $O_k f(x)$ and the operator $O_q f(x), \ q \neq k$ act on different variables in the $f(x)$, that is, these operators are commutative: $O_k O_q f(x) = O_q O_k f(x)$, as well as $O_q O_q f(x) = O_q f(x)$.

Theorem 1 is proved.

Further, an explicit representation of the operator O in terms of the operators $O_1, ..., O_r$ is required without using the identity operator.

Let us recall that in case $r = 1$ the operator $O_1 f(x) = \sum_{i_1=0}^{N} f\left(\frac{i_1}{N}\right) \cdot h\left(N x_1 - i_1\right)$ uses only numbers for its construction – the values of the function $f\left(\frac{i_1}{N}\right)$: $O_1 f\left(\frac{p_1}{N}\right) = f\left(\frac{p_1}{N}\right), \ 0 \leq p_1 \leq N$.

In case $r = 2$ the operators O_1, O_2 use the traces of function $f\left(x_1, x_2\right)$ on the line segments perpendicular to the sides of the square, which lie on the axes Ox_1, Ox_2 respectively:

$$O_1 f\left(x_1, x_2\right) = \sum_{i_1=0}^{N} f\left(\frac{i_1}{N}, x_2\right) \cdot h\left(N x_1 - i_1\right),$$

$$O_2 f\left(x_1, x_2\right) = \sum_{i_2=0}^{N} f\left(x_1, \frac{i_2}{N}\right) \cdot h\left(N x_2 - i_2\right).$$

At the same time, the operator

$$O_1 O_2 f\left(x_1, x_2\right) = \sum_{i_1=0}^{N} \sum_{i_2=0}^{N} f\left(\frac{i_1}{N}, \frac{i_2}{N}\right) \cdot h\left(N \cdot x_1 - i_1\right) \cdot h\left(N \cdot x_2 - i_2\right)$$

is classical two-dimensional spline interpolation operator [4, 5, 15]:

$$O_1 O_2 f\left(\frac{p_1}{N}, \frac{p_2}{N}\right) = f\left(\frac{p_1}{N}, \frac{p_2}{N}\right), \ 0 \leq p_1, p_2 \leq N.$$

In case $r=2$ $x=(x_1,x_2)$ it follows from formula (3) that

$$Of(x)=\left(O_1+O_2-O_1O_2\right)f(x). \tag{4}$$

In case $r=3$, $x=(x_1,x_2,x_3)$ based on formula (3), we can write

$$Of(x)=\left(O_1+O_2+O_3-O_1O_2-O_1O_3-O_2O_3+O_1O_2O_3\right)f(x). \tag{5}$$

Let us note that the term $O_1O_2O_3f(x)$ in formula (5) is the classical formula for spline interpolation of the function of three variables [5, 6].

$$O_1O_2O_3f\left(x_1,x_2,x_3\right)= \\ =\sum_{i_1=0}^{N}\sum_{i_2=0}^{N}\sum_{i_3=0}^{N}f\left(\frac{i_1}{N},\frac{i_2}{N},\frac{i_3}{N}\right)\cdot h\left(Nx_1-i_1\right)\cdot h\left(Nx_2-i_2\right)\cdot h\left(Nx_3-i_3\right),$$

$$O_1O_2O_3f\left(\frac{p_1}{N},\frac{p_2}{N},\frac{p_3}{N}\right)=f\left(\frac{p_1}{N},\frac{p_2}{N},\frac{p_3}{N}\right),\ 0\le p_1,p_2,p_3\le N.$$

Formulas (4) and (5) have the following feature:

In these formulas, the first group of terms is the sum of operators $O_k, k=\overline{1,r}$, the second sum of terms is the sum of products of operators taken two by two with the appropriate sign at $r=2,3$, and the third group of terms is the product of three different operators at $r=3$.

It is obvious that for $r>3$, the last group of terms will be the monomial, which has the form of the product of all operators r with the corresponding sign [3, 9].

From the point of view of applications, we prefer this representation of the operator O and in the next section of the paper we propose a general method of representing the operator O for any natural number r.

3. REPRESENTATION OF OPERATOR O THROUGH OPERATORS $O_1,...,O_r$ WITHOUT USING THE IDENTITY OPERATOR FOR $r\in\Gamma$

In the expression for the operator O: $Of(x)=\left(I-\prod_{k=1}^{r}\left(I-O_k\right)\right)f(x)$ let us replace I by one, O_k by t_k, $k=\overline{1,r}$, and denote the resulting function by $T(t)$.

$$O(t)=T(t)=1-\prod_{k=1}^{r}\left(1-t_k\right)$$

Let us introduce the notation $s=(s_1,...,s_r),\ |s|=s_1+...+s_r$.

$$T_{|s|}(t) = \frac{\partial^{|s|}}{\partial t_1^{s_1} \ldots \partial t_r^{s_r}} T(t) \Bigg|_{t=0} \cdot \prod_{k=1}^{r} t_k^{s_k}.$$

Theorem 2. For each $r \in \Gamma$, $(r \geq 2)$ function $T(t)$ can be represented as follows

$$T(t) = T(0) + \sum_{|s|=1}^{r} T_{|s|}(t). \tag{6}$$

Proof.

Note that is $T(t)$ is r linear function of the variables $t_1, t_2, \ldots, t_r$, that is, it is a polynomial of r from these variables. In this case

$$T(t) = T(0) + \sum_{|s|=1}^{r} t_1^{s_1} .. t_r^{s_r},$$

where $s_k = 0$ or $s_k = 1$. Foe example, at $|s| = 1$ $T_{|s|}(t)$ it is as follows

$$T_{|s|}(t) = \frac{\partial}{\partial t_1} T(t) \Bigg|_{t=0} t_1 + \ldots + \frac{\partial}{\partial t_r} T(t) \Bigg|_{t=0} t_r.$$

Since $|s| = 1$ only for such sets s_k, where one component is equal to one, all the others are zero. Similarly, if $|s| = 2$ only two components are equal to one, all others are zero. For $|s| = r$, all s_k are equal to one, because $T(t)$ has derivatives with respect to each variable t_k of order s_k equal to zero if $2 \leq s_k \leq r$.

This means that $T(t)$ can be exactly represented by the Taylor series expansion of the function around the point $t = 0$, that is $t_k = 0,\ k = \overline{1, r}$, in powers t, which we express using derivatives – described in detail in [1]. It should be noted that $T(t)$ is a polynomial of degree r, each term of which is linear function with respect to each of the arguments $t_1, \ldots, t_r$. This means that the order of partial derivatives s_k with respect to the variable t_k can take the values one or zero.

Taylor formula in terms of derivatives can be written as follows [1]:

$$T(t) = T(0) + \sum_{|s|=1}^{r} T_{|s|}(t). \tag{7}$$

Theorem 2 is proved.

Let's write down the expressions $T_{|s|}(t)$ for $|s| = \overline{1, 5}$.

$$T_{|s|}(t)=\sum_{p_1=1}^{r-|s|+1}\left.\frac{\partial T(t)}{\partial t_{p_1}}\right|_{t=0}\cdot t_{p_1},|s|=1,$$

$$T_{|s|}(t)=\sum_{p_1=1}^{r-|s|+1}\sum_{p_2=p_1+1}^{r}\left.\frac{\partial^{|s|}T(t)}{\partial t_{p_1}\partial t_{p_2}}\right|_{t=0}\cdot t_{p_1}t_{p_2},|s|=2,$$

$$T_{|s|}(t)=\sum_{p_1=1}^{r-|s|+1}\sum_{p_2=p_1+1}^{r-|s|+2}\sum_{p_3=p_2+1}^{r-|s|+3}\left.\frac{\partial^{|s|}T(t)}{\partial t_{p_1}\partial t_{p_2}\partial t_{p_3}}\right|_{t=0}\cdot t_{p_1}t_{p_2}t_{p_3},|s|=3,$$

$$T_{|s|}(t)=\sum_{p_1=1}^{r-|s|+1}\sum_{p_2=p_1+1}^{r-|s|+2}\sum_{p_3=p_2+1}^{r-|s|+3}\sum_{p_4=p_3+1}^{r-|s|+4}\left.\frac{\partial^{|s|}T(t)}{\partial t_{p_1}\partial t_{p_2}\partial t_{p_3}\partial t_{p_4}}\right|_{t=0}\cdot t_{p_1}t_{p_2}t_{p_3}t_{p_4},|s|=4,$$

$$T_{|s|}(t)=\sum_{p_1=1}^{r-|s|+1}\sum_{p_2=p_1+1}^{r-|s|+2}\sum_{p_3=p_2+1}^{r-|s|+3}\sum_{p_4=p_3+1}^{r-|s|+4}\sum_{p_5=p_4+1}^{r-|s|+5}\left.\frac{\partial^{|s|}T(t)}{\partial t_{p_1}\partial t_{p_2}\partial t_{p_3}\partial t_{p_4}\partial t_{p_5}}\right|_{t=0}\cdot t_{p_1}t_{p_2}t_{p_3}t_{p_4}t_{p_5},\ |s|=5.$$

To derive the formula for arbitrary $|s|$, one can use the method of mathematical induction.

Let us perform the inverse substitution t_k for O_k $\left(k=\overline{1,r}\right)$ in formula (7). Then

$$O\left(O_1,...,O_r\right)=T\left(O_1,...,O_r\right)=T(0)+\sum_{|s|=1}^{r}T_{|s|}\left(O_1,...,O_r\right).$$

Let us write expressions for $T_{|s|}\left(O_1,...,O_r\right)$ for $|s|=\overline{1,5}$.

$$T_{|s|}\left(O_1,...,O_r\right)=\sum_{p_1=1}^{r-|s|+1}\left.\frac{\partial T(t)}{\partial t_{p_1}}\right|_{t=0}\cdot O_{p_1},\ |s|=1,$$

$$T_{|s|}\left(O_1,...,O_r\right)=\sum_{p_1=1}^{r-|s|+1}\sum_{p_2=p_1+1}^{r}\left.\frac{\partial^{|s|}T(t)}{\partial t_{p_1}\partial t_{p_2}}\right|_{t=0}\cdot O_{p_1}O_{p_2},\ |s|=2,$$

$$T_{|s|}\left(O_1,...,O_r\right)=\sum_{p_1=1}^{r-|s|+1}\sum_{p_2=p_1+1}^{r-|s|+2}\sum_{p_3=p_2+1}^{r-|s|+3}\left.\frac{\partial^{|s|}T(t)}{\partial t_{p_1}\partial t_{p_2}\partial t_{p_3}}\right|_{t=0}\cdot O_{p_1}O_{p_2}O_{p_3},\ |s|=3,$$

$$T_{|s|}\left(O_1,...,O_r\right)=\sum_{p_1=1}^{r-|s|+1}\sum_{p_2=p_1+1}^{r-|s|+2}\sum_{p_3=p_2+1}^{r-|s|+3}\sum_{p_4=p_3+1}^{r-|s|+4}\left.\frac{\partial^{|s|}T(t)}{\partial t_{p_1}\partial t_{p_2}\partial t_{p_3}\partial t_{p_4}}\right|_{t=0}\cdot O_{p_1}O_{p_2}O_{p_3}O_{p_4},\ |s|=4,$$

$$T_{|s|}\left(O_1,...,O_r\right)=$$
$$=\sum_{p_1=1}^{r-|s|+1}\sum_{p_2=p_1+1}^{r-|s|+2}\sum_{p_3=p_2+1}^{r-|s|+3}\sum_{p_4=p_3+1}^{r-|s|+4}\sum_{p_5=p_4+1}^{r-|s|+5}\left.\frac{\partial^{|s|}T(t)}{\partial t_{p_1}\partial t_{p_2}\partial t_{p_3}\partial t_{p_4}\partial t_{p_5}}\right|_{t=0}\cdot O_{p_1}O_{p_2}O_{p_3}O_{p_4}O_{p_5},\ |s|=5.$$

and so on.

So, we obtained the explicit representation of the operator O through $O_1,...,O_r$ without using the identity operator.

It should be also noted that for $r=2$ and $r=3$ we obtain the explicit formulas (4) and (5) for the operator O, namely

$$r=2;\ Of(x)=\left(\sum_{p_1=1}^{r}O_{p_1}-\left(\sum_{p_1=1}^{r-1}\sum_{p_2=p_1+1}^{r}O_{p_1}O_{p_2}\right)\right)f(x)=\left(O_1+O_2-O_1O_2\right)f(x).$$

$$r=3;\ Of(x)=\left(\sum_{p_1=1}^{r}O_{p_1}-\left(\sum_{p_1=1}^{r-1}\sum_{p_2=p_1+1}^{r}O_{p_1}O_{p_2}\right)+\left(\sum_{p_1=1}^{r-2}\sum_{p_2=p_1+1}^{r-1}\sum_{p_3=p_2+1}^{r}O_{p_1}O_{p_2}O_{p_3}\right)\right)f(x)=$$
$$=\left(O_1+O_2+O_3-O_1O_2-O_1O_3-O_2O_3+O_1O_2O_3\right)f(x).$$

Next, we present expressions for the operator O at $r=4$ and $r=5$.

$$r=4;\ Of(x)=\left(\sum_{p_1=1}^{r}O_{p_1}-\left(\sum_{p_1=1}^{r-1}\sum_{p_2=p_1+1}^{r}O_{p_1}O_{p_2}\right)+\left(\sum_{p_1=1}^{r-2}\sum_{p_2=p_1+1}^{r-1}\sum_{p_3=p_2+1}^{r}O_{p_1}O_{p_2}O_{p_3}\right)-\right.$$
$$\left.-\left(\sum_{p_1=1}^{r-3}\sum_{p_2=p_1+1}^{r-2}\sum_{p_3=p_2+1}^{r-1}\sum_{p_4=p_3+1}^{r}O_{p_1}O_{p_2}O_{p_3}O_{p_4}\right)\right)f(x).$$

$$r=5;\ Of(x)=\left(\sum_{p_1=1}^{r}O_{p_1}-\left(\sum_{p_1=1}^{r-1}\sum_{p_2=p_1+1}^{r}O_{p_1}O_{p_2}\right)+\left(\sum_{p_1=1}^{r-2}\sum_{p_2=p_1+1}^{r-1}\sum_{p_3=p_2+1}^{r}O_{p_1}O_{p_2}O_{p_3}\right)-\right.$$
$$-\left(\sum_{p_1=1}^{r-3}\sum_{p_2=p_1+1}^{r-2}\sum_{p_3=p_2+1}^{r-1}\sum_{p_4=p_3+1}^{r}O_{p_1}O_{p_2}O_{p_3}O_{p_4}\right)+$$
$$\left.+\left(\sum_{p_1=1}^{r-4}\sum_{p_2=p_1+1}^{r-3}\sum_{p_3=p_2+1}^{r-2}\sum_{p_4=p_3+1}^{r-1}\sum_{p_5=p_4+1}^{r}O_{p_1}O_{p_2}O_{p_3}O_{p_4}O_{p_5}\right)\right)f(x).$$

4. INVESTIGATION OF THE REMAINDER OF THE DIFFERENTIAL FUNCTIONS APPROXIMATION BY CLASSICAL SPLINES OF *r* VARIABLES ”

Theorem 3. Remainder $Rf(x)=f(x)-Of(x)$ can be represented as

$$Rf(x) = \left(\prod_{k=1}^{r} R_k \right) f(x) \quad (8)$$

R_k is the product of one-dimensional remainders along x_k, $R_k = I - O_k$.

Proof.

$$Rf(x) = (I - O) f(x) = \left(I - \left(I - \prod_{k=1}^{r} (I - O_k) \right) f(x) \right) =$$
$$= \left(\prod_{k=1}^{r} (I - O_k) \right) f(x) = \left(\prod_{k=1}^{r} R_k \right) f(x).$$

Theorem 3 is proved.

Below we obtain an expression for the remainder of the approximation of the function $f(x)$ by classical splines of r variables by через $R_k, k = \overline{1,r}$.

Let us take into account that

$$R_{Spclasic} f(x) = \left(I - \prod_{k=1}^{r} O_k \right) f(x) = \left(I - \prod_{k=1}^{r} \left(I - (I - O_k) \right) \right) f(x) =$$
$$= \left(I - \prod_{k=1}^{r} (I - R_k) \right) f(x).$$

In this expression let us replace I by one, R_k by $v_k, k = \overline{1,r}$, and denote the resulting function by $T(v)$. Then

$$R_{Spclasic}(v) = T(v) = 1 - \prod_{k=1}^{r} (1 - v_k).$$

Thus, we obtain explicit formulas for the approximation error of the function $f(x)$ by classical spline interpolation operators, expressed in terms of the operators R_k, from the Taylor formula written in terms of derivatives. Then, for the remainder of the approximation $R_{Spclasic} f(x)$ we can use the same approach as in deriving the above mentione formulas for the operator O.

$$T(v) = T(0) + \sum_{|s|=1}^{r} T_{|s|}(v).$$

$$T_{|s|}(v) = \sum_{p_1=1}^{r-|s|+1} \left. \frac{\partial T(v)}{\partial v_{p_1}} \right|_{v=0} \cdot v_{p_1}, |s| = 1,$$

$$T_{|s|}(v) = \sum_{p_1=1}^{r-|s|+1} \sum_{p_2=p_1+1}^{r} \left. \frac{\partial^{|s|} T(v)}{\partial v_{p_1} \partial v_{p_2}} \right|_{t=0} \cdot v_{p_1} v_{p_2}, |s| = 2,$$

$$T_{|s|}(v) = \sum_{p_1=1}^{r-|s|+1} \sum_{p_2=p_1+1}^{r-|s|+2} \sum_{p_3=p_2+1}^{r-|s|+3} \left. \frac{\partial^{|s|} T(v)}{\partial v_{p_1} \partial v_{p_2} \partial v_{p_3}} \right|_{t=0} \cdot v_{p_1} v_{p_2} v_{p_3}, |s| = 3,$$

$$T_{|s|}(v) = \sum_{p_1=1}^{r-|s|+1} \sum_{p_2=p_1+1}^{r-|s|+2} \sum_{p_3=p_2+1}^{r-|s|+3} \sum_{p_4=p_3+1}^{r-|s|+4} \left. \frac{\partial^{|s|} T(v)}{\partial v_{p_1} \partial v_{p_2} \partial v_{p_3} \partial v_{p_4}} \right|_{t=0} \cdot v_{p_1} v_{p_2} v_{p_3} v_{p_4}, |s| = 4,$$

$$T_{|s|}(v) = \sum_{p_1=1}^{r-|s|+1} \sum_{p_2=p_1+1}^{r-|s|+2} \sum_{p_3=p_2+1}^{r-|s|+3} \sum_{p_4=p_3+1}^{r-|s|+4} \sum_{p_5=p_4+1}^{r-|s|+5} \left. \frac{\partial^{|s|} T(v)}{\partial v_{p_1} \partial v_{p_2} \partial v_{p_3} \partial v_{p_4} \partial v_{p_5}} \right|_{t=0} \cdot v_{p_1} v_{p_2} v_{p_3} v_{p_4} t_{p_5}, \ |s| = 5.$$

When deriving the formula for arbitrary $|s|$, one can use the method of mathematical induction.

Let us do the reverse substitution. Instead v_k we will write $R_k, \ k = \overline{1,r}$. Then

$$R_{Spclasic}\left(R_1,...,R_r\right) = T\left(R_1,...,R_r\right) = T(0) + \sum_{|s|=1}^{r} T_{|s|}\left(R_1,...,R_r\right).$$

$$T_{|s|}\left(R_1,...,R_r\right) = \sum_{p_1=1}^{r-|s|+1} \left. \frac{\partial T(v)}{\partial v_{p_1}} \right|_{v=0} \cdot R_{p_1}, \ |s| = 1,$$

$$T_{|s|}\left(R_1,...,R_r\right) = \sum_{p_1=1}^{r-|s|+1} \sum_{p_2=p_1+1}^{r} \left. \frac{\partial^{|s|} T(v)}{\partial v_{p_1} \partial v_{p_2}} \right|_{v=0} \cdot R_{p_1} R_{p_2}, \ |s| = 2,$$

$$T_{|s|}\left(R_1,...,R_r\right) = \sum_{p_1=1}^{r-|s|+1} \sum_{p_2=p_1+1}^{r-|s|+2} \sum_{p_3=p_2+1}^{r-|s|+3} \left. \frac{\partial^{|s|} T(v)}{\partial v_{p_1} \partial v_{p_2} \partial v_{p_3}} \right|_{v=0} \cdot R_{p_1} R_{p_2} R_{p_3}, \ |s| = 3,$$

$$T_{|s|}\left(R_1,...,R_r\right) = \sum_{p_1=1}^{r-|s|+1} \sum_{p_2=p_1+1}^{r-|s|+2} \sum_{p_3=p_2+1}^{r-|s|+3} \sum_{p_4=p_3+1}^{r-|s|+4} \left. \frac{\partial^{|s|} T(v)}{\partial v_{p_1} \partial v_{p_2} \partial v_{p_3} \partial v_{p_4}} \right|_{v=0} \cdot R_{p_1} R_{p_2} R_{p_3} R_{p_4}, \ |s| = 4,$$

$$T_{|s|}\left(R_1,...,R_r\right) = \\ = \sum_{p_1=1}^{r-|s|+1} \sum_{p_2=p_1+1}^{r-|s|+2} \sum_{p_3=p_2+1}^{r-|s|+3} \sum_{p_4=p_3+1}^{r-|s|+4} \sum_{p_5=p_4+1}^{r-|s|+5} \left. \frac{\partial^{|s|} T(v)}{\partial v_{p_1} \partial v_{p_2} \partial v_{p_3} \partial v_{p_4} \partial v_{p_5}} \right|_{v=0} \cdot R_{p_1} R_{p_2} R_{p_3} R_{p_4} R_{p_5}, \ |s| = 5.$$

and so on.

So, we have obtained an explicit representation of the operator $R_{Spclasic}$ through $R_1,...,R_r$ without using the identity operator.

Let us illustrate the above mentioned statements for $r=2$ and $r=3$.

$$r=2;\ R_{Spclassc}f(x)=\left(I-\left(I-R_1\right)\left(I-R_2\right)\right)f(x)=$$
$$=\left(R_1+R_2-R_1R_2\right)f(x)=\left(\sum_{p_1=1}^{r}R_{p_1}-\left(\sum_{p_1=1}^{r-1}\sum_{p_2=p_1+1}^{r}R_{p_1}R_{p_2}\right)\right)f(x).$$

$$r=3;\ R_{Spclasic}f(x)=\left(I-\left(I-R_1\right)\left(I-R_2\right)\left(I-R_3\right)\right)f(x)=$$
$$=\left(R_1+R_2+R_3-R_1R_2-R_1R_3-R_2R_3+R_1R_2R_3\right)f(x)=$$
$$=\left(\sum_{p_1=1}^{r}R_{p_1}-\left(\sum_{p_1=1}^{r-1}\sum_{p_2=p_1+1}^{r}R_{p_1}R_{p_2}\right)+\left(\sum_{p_1=1}^{r-2}\sum_{p_2=p_1+1}^{r-1}\sum_{p_3=p_2+1}^{r}R_{p_1}R_{p_2}R_{p_3}\right)\right)f(x).$$

Next, we present expressions for the operator $R_{Spclasic}$ at $r=4$ and $r=5$.

$$r=4;\ R_{Spclasic}f(x)=\left(I-\left(I-R_1\right)\left(I-R_2\right)\left(I-R_3\right)\left(I-R_4\right)\right)f(x)=$$
$$=\left(\sum_{p_1=1}^{r}R_{p_1}-\left(\sum_{p_1=1}^{r-1}\sum_{p_2=p_1+1}^{r}R_{p_1}R_{p_2}\right)+\left(\sum_{p_1=1}^{r-2}\sum_{p_2=p_1+1}^{r-1}\sum_{p_3=p_2+1}^{r}R_{p_1}R_{p_2}R_{p_3}\right)-\right.$$
$$\left.-\left(\sum_{p_1=1}^{r-3}\sum_{p_2=p_1+1}^{r-2}\sum_{p_3=p_2+1}^{r-1}\sum_{p_4=p_3+1}^{r}R_{p_1}R_{p_2}R_{p_3}R_{p_4}\right)\right)f(x).$$

$$r=5;\ R_{Spclasic}f(x)=\left(I-\left(I-R_1\right)\left(I-R_2\right)\left(I-R_3\right)\left(I-R_4\right)\left(I-R_5\right)\right)f(x)=$$
$$=\left(\sum_{p_1=1}^{r}R_{p_1}-\left(\sum_{p_1=1}^{r-1}\sum_{p_2=p_1+1}^{r}R_{p_1}R_{p_2}\right)+\left(\sum_{p_1=1}^{r-2}\sum_{p_2=p_1+1}^{r-1}\sum_{p_3=p_2+1}^{r}R_{p_1}R_{p_2}R_{p_3}\right)-\right.$$
$$-\left(\sum_{p_1=1}^{r-3}\sum_{p_2=p_1+1}^{r-2}\sum_{p_3=p_2+1}^{r-1}\sum_{p_4=p_3+1}^{r}R_{p_1}R_{p_2}R_{p_3}R_{p_4}\right)+$$
$$\left.+\left(\sum_{p_1=1}^{r-4}\sum_{p_2=p_1+1}^{r-3}\sum_{p_3=p_2+1}^{r-2}\sum_{p_4=p_3+1}^{r-1}\sum_{p_5=p_4+1}^{r}R_{p_1}R_{p_2}R_{p_3}R_{p_4}R_{p_5}\right)\right)f(x).$$

It should be noted that at small values R_k for remainder $R_{Spclasic}$, the asymptotic relation can be written – its application in physical and technical problems is described

$$R_{Spclasic}=\sum_{k=1}^{r}R_k,\ N>>1$$
.

5. COMPARISON OF ERROR ESTIMATES WHEN APPROXIMATING THE DIFFERENTIAL FUNCTION OF *r* VARIABLES BY SPLINE INTERFLATATION OPERATORS AND CLASSICAL SPLINE INTERPOLATION

In order to prove the following theorem, we need formulas for estimating the errors of one-dimensional spline interpolation (first-degree splines) with respect to the variable u – the application in physical and technical problems is described in [1, 12].

For function

$$g(u)\in C^2[a,b]\Rightarrow \left|R_n(u)=g(u)-s_{n,1}(u)\right|\le \frac{\Delta^2 M_2}{8};$$

$$M_2=\max_{u\in[a,b]}\left|g''(u)\right|,\ \Delta=\max\left(u_m-u_{m-1}\right),\ 1\le m\le n;$$

$$s_{n,1}(u)=g\left(u_{m-1}\right)\frac{u-u_m}{u_{m-1}-u_m}+g\left(u_m\right)\frac{u-u_{m-1}}{u_m-u_{m-1}},\ u_{m-1}\le u\le u_m,\ m=\overline{1,n}$$

$s_{n,1}(u)$ is first degree spline (piecewise linear spline).

In the case of uniform division of the segment, we will assume $\Delta=\frac{1}{N}$.

Then

$$\left|R_n(u)\right|\le \frac{M_2}{2^3 N^2}. \tag{9}$$

Theorem 4. For any function $f(x)\in C^{2,\dots,2}(D)$, the following inequality is true

$$\left|Rf(x)\right|=\left|\left(\prod_{k=1}^{r}R_k\right)f(x)\right|\le \frac{M_2^r}{2^{3r}N^{2r}}. \tag{10}$$

Proof.

Taking into account formula (7), which expresses the remainder of the approximation of the function $f(x)$ as the product of the remainders for each of r variables, and inequality (8), we obtain formula (9).

Theorem 4 is proved.

Considering the above mentioned, we have $\left|Rf(x)\right|\le \frac{M_2^r}{2^{3r}N^{2r}}$ and $\left|R_{Spclasic}f(x)\right|\le r\frac{M_2}{2^3N^2}$.

In order for classical spline interpolation to provide the same order of accuracy with respect to the variable N, as the operator O, it is necessary in classical interpolation to replace N with N^r in all r sums. This requires the use of N^{2r} values of the approximating function. So, at $N>>1$ $\left|Rf(x)\right|\le c_1N^{-2r}$, $\left|R_{Spclasic}f(x)\right|\le c_2N^{-2},c_1,c_2-const$.

6. CONCLUSIONS

The paper investigates The relationship between two types of approximation operators for differential functions r of variables $(r \geq 2)$ is investigated in this paper. The first type consists of spline interflatation operators, which are constructed using information about the function being approximated in the form of spline interpolation O_k operators on each edge of the unit cube separately. The second type consists of classical interpolation operators, which use information about the function being approximated simultaneously on all edges. Despite the fact that these operators are known, the relationship between them and their approximation remainders has not been established in the general formulation.

The paper formulates and proves that there is a duality between these types of operators. In the first type, the approximation remainder is the operator product of the remainders of one-dimensional approximation operators R_k. Whereas in the second type, the approximation operator is the product of one-dimensional approximation operators O_k..

Moreover, it is proven that the formula for the explicit representation of the first-type operators in terms of the one-dimensional approximation operators O_k used in the second-type approximation operators coincides with the formula for the remainder when O_k is replaced by R_k, which is also proven. Examples are provided for the decomposition of interflatation operators and remainder operators of classical spline interpolation of differential functions of r variables. This can be used for compressing information about differentiable functions of many variables, as follows from the comparison of their computational complexity provided in the work while ensuring the same approximation accuracy.

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ТЕОРІЯ СПЛАЙН-ІНТЕРФЛЕТАЦІЇ ФУНКЦІЙ r ЗМІННИХ r ≥ 2

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Резюме. *Розроблено теорію наближення багатовимірних функцій r змінних* $(r \geq 2)$ *за допомогою операторів сплайн-інтерфлетації. В рамках дослідження запропоновано новий метод побудови таких операторів, що базується на підході розкладу багатовимірної задачі наближення на послідовність одновимірних задач, кожна з яких вирішується за допомогою сплайн-інтерполяції. Це дозволяє досліджувати інтерфлетаційні властивості побудованих операторів, а також аналізувати їхню ефективність у наближенні функцій з кількома змінними. Особливістю запропонованого методу є явне представлення операторів сплайн-інтерфлетації через одновимірні оператори сплайн-інтерполяції, які застосовуються окремо до кожної змінної функції, що наближується. Це забезпечує зручність у дослідженні властивостей операторів і дозволяє глибше аналізувати їхню поведінку. В рамках роботи досліджено вираз залишку наближення функцій за допомогою цих операторів, зокрема, через залишки наближення, що виникають при застосуванні одновимірних операторів сплайн-інтерполяції. Особливу увагу приділено аналізу залишків наближення багатовимірних функцій і доведенню того, що залишок наближення, що обчислюється за допомогою запропонованих операторів інтерфлетації, дорівнює операторному добутку залишків наближення, які визначаються для кожної змінної окремо. Це означає, що повний залишок можна розглядати як комбінацію залишків, отриманих через одновимірні оператори, що значно спрощує аналіз і дає можливість детальніше досліджувати точність наближення. Крім того, проведено порівняльний аналіз отриманих результатів із класичними операторами багатовимірної інтерполяції. Зокрема, зроблено порівняння з операторами класичної інтерполяції, що дозволяє оцінити переваги й недоліки запропонованої методики у контексті точності та ефективності наближення функцій з кількома змінними. Це відкриває перспективи для подальшого розвитку теорії багатовимірного наближення та застосування її в різних галузях науки й техніки, де необхідне ефективне й точне наближення багатовимірних функцій.*

Ключові слова: *сплайн-інтерфлетація, оператор, похибка наближення, інтерполяція, залишок, диференційовна функція, формула Тейлора.*

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