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PRESENTATION OF SOLUTIONS OF THREE-DIMENSIONAL DYNAMIC PROBLEMS OF THE THEORY OF ELASTICITY IN A CURVILINEAR ORTHOGONAL COORDINATE SYSTEM

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Abstract. *To describe the processes of distribution of elastic waves, a model of a three -dimensional isotropic body is used under the action of dynamic loads. A well known presentation of the general solution of equations had been considered in a vector form, which contains four functions. It is established that the function that describes the expansion waves is uniquely determined by the volume* deformation. *It is shown that the dynamic tense-deformed state of the body with zero volumetric extension can be expressed through two independent functions that satisfy the equation that describes the waves of shift. It is proved that the overall solution of equations can be expressed through three four dimensional displacement functions, which are defined as the solutions of wave equations of the second order. This solution was used and an analytical expression of the general solution of the equations of dynamic theory of elasticity in the curvilinear orthogonal coordinate system was found. This submission has been used and a clear expression of elastic displacements in the cylindrical coordinate system was recorded. However, there are multipliers near one displacement function, which depend on the angular variable, which complicates its practical use. The general solution is regulated in the cylindrical coordinate system in such a way that the coefficients of the expansion in the Fourier rows do not depend on the angle* φ. *This made it possible to significantly simplify the expression of solution. The components of a stress-deformed state in the cylindrical coordinate system are recorded.*

Key words: *cylindrical coordinate system, solution of Navier's equations, dynamic stresses and displacements.*

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1. INTRODUCTION

Integrating the equations of elasticity theory and finding stresses in an elastic body under dynamic loads is an important task [1, 2]. The study of the processes of propagation and diffraction of elastic waves in bodies under the action of a time-varying load is based on the use of cumbersome mathematical approaches [1, 3, 4], which use the presentation of the general solution of dynamic equilibrium equations.

Setting and solving problems of elastic dynamics from the beginning of the 19th century. was the object of many studies [5, 6]; in particular, they sought to generalize the solution of G. Lamé's static equations for isotropic, linearly elastic bodies in the dynamic case in the Cartesian coordinate system. These dynamic solutions were originally found by C. L. M.-H. Navier in the partial case when $\lambda = \mu$; and also later G. Lamé [7] and G. G. Stokes [6]. In fluid mechanics, they are known as the Navier-Stokes equations. In 1892, C. Somigliana [8] gave a new representation of the solution of the Navier-Stokes equations using three scalar potential functions, each of which satisfies the double wave equation (fourth order in partial derivatives). This representation remained forgotten for a long time until M. Iacovache [6] independently rediscovered it in vector form in 1949. Based on the representation of Somigliana [8], E. Sternberg and R. A. Eubanks [9] proposed a representation of the solution using four scalar potential functions, one of which satisfies the equation of longitudinal waves (second order in partial derivatives), and the other three satisfy the equation of transverse second order waves. This representation of the Navier-Stokes equations in the dynamic case generalizes the well-known Papkovich-Neuber [2] representation given for the Lamé equations in the static case.

The paper [10] provides a quantitative and qualitative analysis of wave fields in a half-space, a layer, and a cylinder. In [10], the peculiarities of the reflection of elastic waves from the boundary of these regions were considered, and in [11], the scattering of waves in a half-space was investigated and the peculiarities of its elastic wave motions were studied. The problems of the dynamic theory of elasticity are among the most complex problems of a deformable solid. Currently, representations of the general solution in vector form are widely used [1, 4, 5, 10], which contain four functions that satisfy second-order wave equations.

The work [12] proposed a method of reducing dynamic problems of the theory of elasticity for an infinite body weakened by a crack to integral singular equations using four potentials. It was shown that it is sufficient to consider only three potential functions. In work [13] it is proved that the general solution of the equations of the static isotropic theory of elasticity is expressed in terms of three three-dimensional harmonic functions that satisfy the static equations of the second order. The purpose of the article is to find a general solution of three-dimensional dynamic problems of the theory of elasticity through three functions in Cartesian, curvilinear orthogonal and cylindrical coordinate systems.

2. FORMULATION OF THE PROBLEM AND RECORDING OF EQUILIBRIUM EQUATIONS OF AN ELASTIC BODY IN THE DYNAMIC CASE

Consider the general formulation of a three-dimensional dynamic problem of elasticity theory. Suppose an elastic isotropic body is in a state of elastic dynamic equilibrium: $x_1 = x,\ x_2 = y,\ x_3 = z$. We will assume that the dynamic deformations of the body are negligible and depend on time in a nonlinear manner.

Strain components according to Hooke's law are related to stress components [2, 5]:

$$\varepsilon_j = \gamma_{jj} = \frac{1}{E}\{(1+\nu)\sigma_j - \nu\Theta\}, \quad \gamma_{kj} = \frac{1}{G}\tau_{kj}, k \neq j, \tag{1}$$

where E is Young's modulus; ν is Poisson's ratio; $\Theta = \sigma_1 + \sigma_2 + \sigma_3$, $G = \dfrac{E}{2(1+\nu)}$ is the shear modulus. We also write down the explicit expression of the physical relations

$$\tau_{kk} \equiv \sigma_k = \lambda e + 2G\varepsilon_k, \quad \tau_{kj} = G\gamma_{kj}, k \neq j, \tag{2}$$

where $\lambda = \dfrac{\nu E}{(1-2\nu)(1+\nu)}$, $e = div\,\mathbf{u} = \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} + \dfrac{\partial u_3}{\partial x_3}$ is a volume deformation. Tensile and shear deformations are as follows

$$\gamma_{jj} \equiv \varepsilon_j = \frac{\partial u_j}{\partial x_j}, \quad j = \overline{1,3}, \ \gamma_{kj} = \frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k}, \ k, j = \overline{1,3}, \ k \neq j. \tag{3}$$

We substitute the components of the stress tensor (2) into the equilibrium equation, take into account the expression of deformations (3), and after transformations we obtain the known equations of dynamic equilibrium in displacements [2, 4]

$$G\nabla^2\mathbf{u} + (G+\lambda)grad\, e = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}. \tag{4}$$

Applying the divergence operator to the system of equations (4), we obtain the equation that must satisfy the volume deformation

$$L_1 e = 0, \tag{5}$$

where $L_1 = c_1^2\nabla^2 - \frac{\partial^2}{\partial t^2}$ is the wave operator, $c_1 = \sqrt{\frac{2G(1-\nu)}{(1-2\nu)\rho}}$ is the speed of propagation of expansion waves.

It is known [2, 4] that the volume deformation can be expressed in terms of a single function

$$\mathbf{u} = grad\,\Phi,\ e = \nabla^2\Phi, \tag{6}$$

where the function Φ satisfies equation (5). In [1, 4], a general solution to the system of equations (4) is given through four functions

$$\mathbf{u} = grad\,\Phi + rot\,\Psi_1\mathbf{i} + rot\,\Psi_2\mathbf{j} + rot\,\Psi_3\mathbf{k}, \tag{7}$$

where the functions Ψ_j, $j=\overline{1,3}$ satisfy the equations

$$L_2\Psi_j = 0,\ j=\overline{1,3},\ L_2 = c_2^2\nabla^2 - \frac{\partial^2}{\partial t^2}, \tag{8}$$

$c_2 = \sqrt{G/\rho}$ is the propagation speed of shear waves. It is known [3, 5], that a part of the vector rotational displacements (7) $\mathbf{u_j} = rot(\Psi_j \mathrm{e}_\mathbf{j})$, $j=\overline{1,3}$ have zero volume deformation. Therefore, the function Φ in form (7) is certainly determined by the volume deformation

$$e = c_1^{-2}\frac{\partial^2\Phi}{\partial t^2},\ \Phi = c_1^2\iint e\,dtdt. \tag{9}$$

The unknown integration functions in the second equation (9) can be set to zero, since the displacements are nonlinear in time.

3. CONSTRUCTION OF THE GENERAL SOLUTION OF THE EQUATIONS OF THE DYNAMIC THEORY OF ELASTICITY THROUGH THREE FUNCTIONS

Let us find the general expression of elastic displacements that satisfy equation (4) but have zero volume deformation, $e=0$. For such displacements, the system of equations (4) will take a simplified form

$$L_2 u_k = 0,\ k=\overline{1,3}. \tag{10}$$

The condition $e = 0$ implies that the displacements (10) have to satisfy the equation:

$$\frac{\partial u_1}{\partial x_1} = -\frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3}. \tag{11}$$

Lemma 1. *The dynamic stress-strain state of a body with zero volume deformation can be given as*

$$\mathbf{u} = rot\,\Psi_1 \mathbf{j} + rot\,\Psi_2 \mathbf{k}, \tag{12}$$

where Ψ_k, $k = \overline{1,2}$ *are independent functions satisfying equation (8).*

Proof. The vector displacements (12) have the form

$$u_1 = \frac{\partial \Psi_2}{\partial y} - \frac{\partial \Psi_1}{\partial z}, \quad u_2 = -\frac{\partial^2 \Psi_2}{\partial x}, \quad u_3 = \frac{\partial \Psi_1}{\partial x}, \tag{13}$$

where $\Psi_k = \frac{\partial \psi_k}{\partial z}$, $k = \overline{1,2}$. The components of the vector (13) satisfy equations (8), (11).

Let us consider an arbitrary vector solution w of equations (8) under condition (11). In general, its components can be represented as

$$\mathrm{w}_1 = \frac{\partial \psi_1}{\partial x}, \quad w_2 = \frac{\partial \psi_2}{\partial x}, \quad w_3 = \frac{\partial \psi_3}{\partial x}, \tag{14}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z}\right) = 0, \tag{15}$$

where ψ_j are functions satisfying equations (8), (11). Let us show that the displacement representations (13), (14) are equivalent. We compare them with each other and obtain equalities: $\Psi_1 = \psi_3$, $\Psi_2 = -\psi_2$. Subtract the components of (13) from the relations (14) and write the displacement difference $v_j = \mathrm{w}_j - u_j$:

$$v_1 = \frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z}, \quad v_2 = 0, \quad v_3 = 0. \tag{16}$$

Since the condition (15) coincides with the first equation (16), the component v_1 is zero. Therefore, representations of the displacements (13), (14) will be equivalent if we put:

$$\psi_3 = \Psi_1, \quad \psi_2 = -\Psi_2, \quad \frac{\partial \psi_1}{\partial x} = \frac{\partial \Psi_2}{\partial y} - \frac{\partial \Psi_1}{\partial z}. \tag{17}$$

The lemma is proved.

Theorem 1. *The general solution of the system of equations (4) can be represented as*

$$\mathbf{u} = grad\,\Phi + rot\,\mathrm{Q}_1 \mathbf{j} + rot\,Q_2 \mathbf{k}, \tag{18}$$

where function $\Phi = c_1^2 \iint e\,dtdt$ *satisfies equation (5), and functions* Q_k, $k=\overline{1,2}$ *satisfy equation (8).*

Proof. We have established that the vector of elastic displacements (18) satisfies equation (4). Let us prove that any solution $\mathbf{u}(x_1, x_2, x_3, t)$ of the system of equations (4) can always be represented in the form (18). If its volume deformation is zero, then according to Lemma 1, it will have the form (18), where $\Phi \equiv 0$. Consider the case when the volume deformation is not zero, $e \neq 0$. According to equation (9), it is always possible to choose a function Φ that defines a given non-zero volume deformation. The theorem is proved.

4. OBTAINING A GENERAL SOLUTION TO THE EQUATIONS OF THE DYNAMIC THEORY OF ELASTICITY IN THE CURVED ORTHOGONAL COORDINATE SYSTEM

To represent the elastic displacements in the curved coordinate system [14], we use the solution (18), which was found in the invariant vector form. In the curved coordinate system $\zeta_j(x_1, x_2, x_3, t)$, $j=\overline{1,3}$, the displacement vector is notated by $u_\zeta(\zeta_1, \zeta_2, \zeta_3, t)$, and according to the vector transformation formulas [14], its components will be

$$u_{\zeta k} = \frac{1}{h_k}\frac{\partial \Phi}{\partial \zeta_k} + Q_k^1 + Q_k^2, \; k=\overline{1,3}, \tag{19}$$

where

$$h_k = \sqrt{\left(\frac{\partial x_1}{\partial \zeta_k}\right)^2 + \left(\frac{\partial x_2}{\partial \zeta_k}\right)^2 + \left(\frac{\partial x_3}{\partial \zeta_k}\right)^2};$$

$$Q_1^j = \frac{1}{h_2 h_3}\left\{\frac{\partial}{\partial \zeta_2}\left[\frac{\partial \zeta_3}{\partial x_m} h_3^2 \Psi_j\right] - \frac{\partial}{\partial \zeta_3}\left[\frac{\partial \zeta_2}{\partial x_m} h_2^2 \Psi_j\right]\right\},$$

$$Q_2^j = \frac{1}{h_1 h_3}\left\{\frac{\partial}{\partial \zeta_3}\left[\frac{\partial \zeta_1}{\partial x_m} h_1^2 \Psi_j\right] - \frac{\partial}{\partial \zeta_1}\left[\frac{\partial \zeta_3}{\partial x_m} h_3^2 \Psi_j\right]\right\}, \tag{20}$$

$$Q_3^j = \frac{1}{h_2 h_1}\left\{\frac{\partial}{\partial \zeta_1}\left[\frac{\partial \zeta_2}{\partial x_m} h_2^2 \Psi_j\right] - \frac{\partial}{\partial \zeta_2}\left[\frac{\partial \zeta_1}{\partial x_m} h_1^2 \Psi_j\right]\right\}, \; j=\overline{1,2}, \; m = j+1$$

are the components of the rotor vector in the orthogonal coordinate system.

The deformations are found after using the displacements (19) by formulas [5]

$$\gamma_{km} = \frac{h_k}{h_m}\frac{\partial}{\partial \zeta_m}\left[\frac{u_{\zeta k}}{h_k}\right] + \frac{h_m}{h_k}\frac{\partial}{\partial \zeta_k}\left[\frac{u_{\zeta m}}{h_m}\right], \quad k \neq m, \quad k,m=\overline{1,3},$$

$$\varepsilon_{kk} = \frac{1}{h_k}\frac{\partial u_{\zeta k}}{\partial \zeta_k} + \sum_{j\neq k}^{3} \frac{1}{h_k h_j}\frac{\partial h_k}{\partial \zeta_j} u_{\zeta j}, \quad k, j = \overline{1,3}. \tag{21}$$

The components of the stress tensor can be determined from the relations (2). The stresses defined by formulas (2), (21) describe the stress-strain state of a three-dimensional body in an orthogonal coordinate system through three independent functions.

5. EXPRESSION OF THE COMPONENT OF THE DYNAMIC STRESS-DEFORMED STATE IN THE CYLINDRICAL COORDINATE SYSTEM

Relations (2), (21) are used to determine the stress tensor in the cylindrical coordinate system $\zeta_1 = r,\ \zeta_2 = \varphi,\ \zeta_3 = z$ $(r \geq 0,\ \varphi \subset [0,2\pi])$ which is related to the Cartesian system by the following formulas [14]

$$x_1 = r\cos\varphi,\ x_2 = r\sin\varphi,\ x_3 = z,\ h_1 = 1, h_2 = r, h_3 = 1,$$
$$\mathbf{i} = \cos\varphi e_r - \sin\varphi e_\varphi,\ \mathbf{j} = \cos\varphi e_\varphi + \sin\varphi e_r. \tag{22}$$

Substitute dependencies (22) into relation (19) and write the expression of elastic displacements in the cylindrical coordinate system

$$u_r = \frac{\partial \Phi}{\partial r} + \frac{1}{r}\frac{\partial \Psi}{\partial \varphi} - \cos\varphi\frac{\partial \Psi_{1r}}{\partial z},\ u_\varphi = \frac{1}{r}\frac{\partial \Phi}{\partial \varphi} - \frac{\partial \Psi}{\partial r} + \sin\varphi\frac{\partial \Psi_{1r}}{\partial z},$$
$$u_z = \frac{\partial \Phi}{\partial z} + \cos\varphi\frac{1}{r}\frac{\partial}{\partial r}(r\Psi_{1r}) - \frac{1}{r}\frac{\partial}{\partial \varphi}(\sin\varphi \Psi_{1r}), \tag{23}$$

where Φ, Ψ, Ψ_{1r} are independent displacement functions that depend on r, φ, z, t. The function Φ satisfies the wave equation (5), and functions Ψ, Ψ_{1r} satisfy equation (8), where

$$\nabla^2 = \nabla_r^2 = [\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}].$$

The part of the solution (23) defined by the function Ψ_{1r} has a complex expression. This is because it involves factors that depend on the angle φ. To construct a simpler expression of this part of the solution, we will use the general solution (7). Let us represent the functions Ψ_1, Ψ_2 from (7) as the following Fourier series with respect to the angle φ [15]:

$$\Psi_1 = \sum_{n=1}^{\infty}(\Psi_n^1 \cos n\varphi + Q_n^1 \sin n\varphi),\ \Psi_2 = \sum_{n=1}^{\infty}(\Psi_n^2 \cos n\varphi + Q_n^2 \sin n\varphi), \tag{24}$$

where the coefficients Ψ_n^j, Q_n^j depend only on the variables r, z, t. Since the functions Ψ_1, Ψ_2 are solutions of equation (8), the coefficients Ψ_n^j, Q_n^j satisfy the equation

$$L_n^2 \Psi_n^j = 0,\ j = \overline{1,2},\ L_n^2 = c_2^2[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{n^2}{r^2} + \frac{\partial^2}{\partial z^2}] - \frac{\partial^2}{\partial t^2}. \tag{25}$$

We put

$$Q_n^1 = \pm\Psi_n^2,\ Q_n^2 = \pm\Psi_n^1. \tag{26}$$

We will use dependencies (24)–(26) and write the part of the rotary solution (7) in the cylindrical coordinate system

$$\begin{aligned}\Psi_1\mathbf{i} + \Psi_2\,\mathbf{j} = \sum_{n=0}^{\infty}\{[\cos(n\mp 1)\varphi\Psi_n^1 + \sin(1\pm n)\varphi\Psi_n^2]\mathrm{e}_r +\\ +[\cos(n\pm 1)\varphi\Psi_n^2 - \sin(1\mp n)\varphi\Psi_n^1]\mathrm{e}_\varphi.\end{aligned} \tag{27}$$

To be clear, let us put a sign «+» in expressions (26) and rewrite the relation (27)

$$\begin{aligned}\Psi_1\mathbf{i} + \Psi_2\,\mathbf{j} = \sum_{n=0}^{\infty}\{[\cos(n-1)\varphi\Psi_n^1 + \sin(n+1)\varphi\Psi_n^2]\mathrm{e}_r + [\cos(n+1)\varphi\Psi_n^2 +\\ +\sin(n-1)\varphi\Psi_n^1]\mathrm{e}_\varphi.\end{aligned} \tag{28}$$

Let us use the representation (28), replace the indices and determine the direct value of the rotor from the vector (28) in the cylindrical coordinate system [14].

$$\begin{aligned}u_r &= -\sum_{n=1}^{\infty}\frac{\partial}{\partial z}[\cos n\varphi\Psi_{n-1}^2 + \sin n\varphi\Psi_{n+1}^1],\\ u_\varphi &= \sum_{n=1}^{\infty}\frac{\partial}{\partial z}[\cos n\varphi\Psi_{n+1}^1 + \sin n\varphi\Psi_{n-1}^2],\\ u_z &= \sum_{n=1}^{\infty}\{\frac{n}{r}[\Psi_{n+1}^1\sin n\varphi - \Psi_{n-1}^2\cos n\varphi] + \frac{\partial}{\partial r}[\Psi_{n-1}^2\cos n\varphi + \Psi_{n+1}^1\sin n\varphi].\end{aligned} \tag{29}$$

It should be noted that the displacements (29) replace the displacements defined by the function Ψ_{1r} in expressions (23). The value of the lower index in the functions is determined by equation (25).

We also represent the functions Φ, Ψ from representation (23) as Fourier series:

$$\Phi = \sum_{n=1}^{\infty}(\Phi_n^1\cos n\varphi + \Phi_n^2\sin n\varphi),\ \ \Psi = \sum_{n=1}^{\infty}(\Psi_n\cos n\varphi + Q_n\sin n\varphi)\ , \tag{30}$$

where coefficients Φ_n^j, Q_n, Ψ_n depend only on variables r, z, t and satisfy equation (25). Substitute the expands (30) into the relations (23) and determine the displacements. Add the displacements (29) to them and we obtain the general expression for the displacements

$$\begin{aligned}u_r &= \sum_{n=1}^{\infty}\{(\frac{\partial\Phi_n^1}{\partial r} + \frac{n}{r}Q_n - \Psi_{n-1}^2)\cos n\varphi + (\frac{\partial\Phi_n^2}{\partial r} - \frac{n}{r}\Psi_n - \Psi_{n+1}^1)\sin n\varphi\},\\ u_\varphi &= \sum_{n=1}^{\infty}\{(\frac{n}{r}\Phi_n^2 - \frac{\partial\Psi_n}{\partial r} + \Psi_{n+1}^1)\cos n\varphi - (\frac{n}{r}\Phi_n^1 + \frac{\partial Q_n}{\partial r} - \Psi_{n-1}^2)\sin n\varphi\},\\ u_z &= \sum_{n=1}^{\infty}\{[\frac{\partial\Phi_n^1}{\partial z} + \frac{\partial}{\partial r}\Psi_{n-1}^2 - \Psi_{n-1}^2]\cos n\varphi + [\frac{\partial\Phi_n^2}{\partial z} + \frac{\partial}{\partial r}\Psi_{n+1}^1 - \frac{n}{r}\Psi_{n+1}^1]\sin n\varphi\}.\end{aligned} \tag{31}$$

We write [5] the expression of the strain of relative elongation and shear in the cylindrical coordinate system

$$\varepsilon_r = \frac{\partial u_r}{\partial r},\ \varepsilon_\varphi = \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_r}{r},\ \ \varepsilon_z = \frac{\partial u_z}{\partial z},$$

$$\gamma_{r\varphi} = r\frac{\partial}{\partial r}\frac{u_\varphi}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \varphi},\ \ \gamma_{rz} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z},\ \gamma_{z\varphi} = \frac{1}{r}\frac{\partial u_z}{\partial \varphi} + \frac{\partial u_\varphi}{\partial z}. \quad (32)$$

Having substituted the displacements (31) into the relation (32), we find the deformations of relative elongation and shear. An important feature of the representation of displacements (31) is that their volume deformation is determined by formula (9). Substitute the obtained strains into dependence (2) and find the stresses.

6. CONCLUSIONS

It is established that the general solution of the equations of the theory of elasticity in an elastic body for dynamic loads can be expressed in terms of three functions, which are defined as solutions of second-order wave equations. A simple expression of the function was found, which describes the propagation of expansion-compression waves due to volume expansion. It is shown that the dynamic stress-strain state of a body with zero volumetric expansion can be expressed in terms of two independent functions that satisfy the second-order equation describing shear waves. The expansion for the solution of the dynamic equations of the theory of elasticity in the cylindrical coordinate system is constructed so that it does not contain factors that include functions that depend on the angle φ. For the first time, displacements and deformations that describe oscillations in a cylindrical coordinate system were found.

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ПОДАННЯ РОЗВ'ЯЗКІВ ТРИВИМІРНИХ ДИНАМІЧНИХ ЗАДАЧ ТЕОРІЇ ПРУЖНОСТІ В КРИВОЛІНІЙНІЙ ОРТОГОНАЛЬНІЙ СИСТЕМІ КООРДИНАТ

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Резюме. *Для описування процесів поширення пружних хвиль застосовано модель тривимірного ізотропного тіла під дією динамічних навантажень. Використано узагальнені співвідношення Гука для подання напружень в однорідному твердому тілі. Після підстановки пружних напружень у динамічні рівняння рівноваги записано систему диференціальних рівнянь Нав'є в частинних похідних другого порядку на пружні переміщення. Розглянуто відоме подання загального розв'язку рівнянь Нав'є у векторному вигляді, яке містить чотири функції. Встановлено, що функція, яка описує хвилі розширення, однозначно визначається через об'ємне розширення. Показано, що динамічний напружено-деформований стан тіла з нульовим об'ємним розширенням можна виразити через дві незалежні функції, які задовольняють рівняння, що описує хвилі зсуву. Розглянуто декартову систему координат і доведено, що загальний розв'язок рівнянь Нав'є можна виразити через три чотирихмірні функції переміщень, які визначаються як розв'язки хвильових рівнянь другого порядку. Використано цей розв'язок і знайдено аналітичний вираз загального розв'язку рівнянь динамічної теорії пружності у криволінійній ортогональній системі координат через три незалежні функції переміщень. Як частковий випадок цього подання записано явний вираз пружних переміщень у циліндричній системі координат. Проте в цьому подані біля однієї функції переміщень стоять множники, які залежать від кутової змінної* φ*, що ускладнює його практичне застосування. Для побудови простішого виразу цього розв'язку використано два загальних розв'язки рівнянь Нав'є в декартовій системі координат. Після переведення їх у циліндричну систему координат функції переміщень розкладено в ряди Фур'є відносно кута* φ*. Проведено регуляризацію загального розв'язку таким чином, що усунуто в коефіцієнтах розкладу члени, які залежать від кута* φ*. Це суттєво спростило вираз розв'язку. Записано компоненти напружено-деформованого стану в циліндричній системі координат.*

Ключові слова: *циліндрична система координат, розв'язок рівнянь Нав'є, динамічні напруження і переміщення.*

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