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Ternopil Ivan Puluj National Technical University

Higher Mathematics
Department

TUTORIAL
FROM THE COURSE OF
HIGHER MATHEMATICS
(Continuity. Derivatives)

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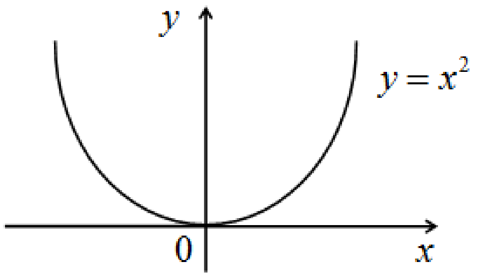
Continuity

Derivatives

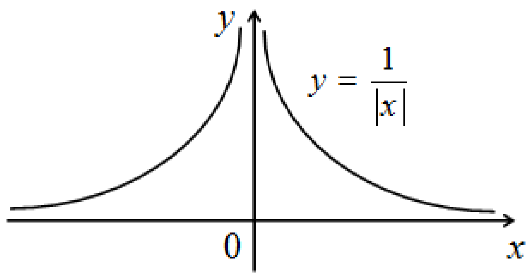
Continuity

1. Introduction

Example 1 $f(x)=x^2$ is continuous on R



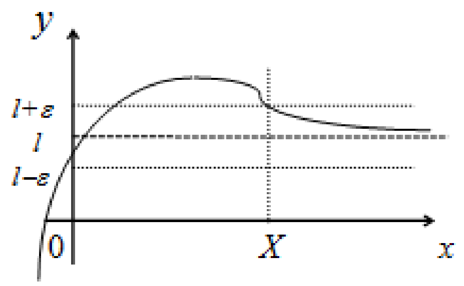
Example 2 $f(x)=\frac{1}{|x|}$ is discontinuous at $x=0$

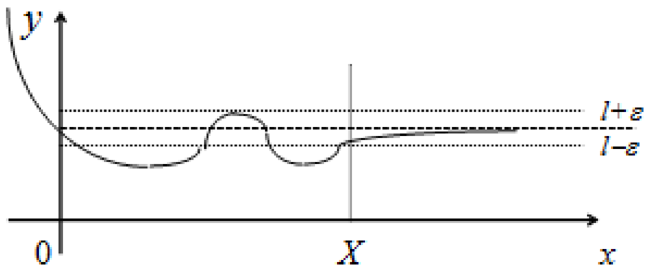


2. Limit of a Function

A. LIMIT OF A FUNCTION AT INFINITY

Defintion Let $f(x)$ be a function defined on R. $\lim_{x\to\infty} f(x)=l$ means that for any $\varepsilon>0$, there exists $X>0$ such that when $x>X$, $|f(x)-l|<\varepsilon$.





N.B.

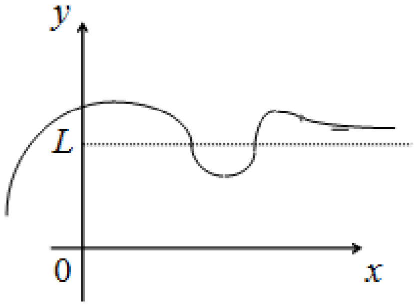
(1) $\lim_{x\to\infty} f(x)=l$ means that the difference between $f(x)$ and l can be made arbitrarily small when x is sufficiently large.

(2) $\lim_{x\to\infty} f(x)=l$ means $f(x)\to A$ as $x\to\infty$.

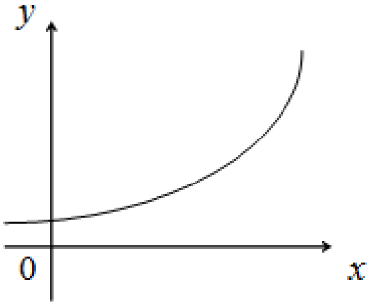
(3) Infinity, ∞, is a symbol but not a real value.

There are three cases for the limit of a function when $x\to\infty$.

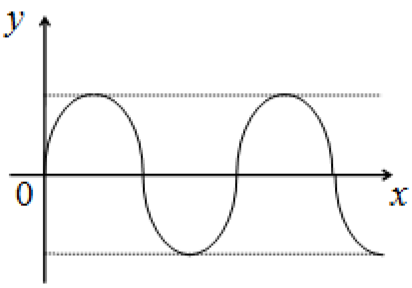
1. $f(x)$ may have a finite limit value.



2. $f(x)$ may approach to infinity.



3. $f(x)$ may oscillate or infinitely and limit value does not exist.



Theorem **UNIQUENESS of Limit Value**

If $\lim\limits_{x\to\infty} f(x) = a$ and $\lim\limits_{x\to\infty} f(x) = b$, then $a = b$.

Theorem **Rules of Operations on Limits**

If $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to\infty} g(x)$ exist and have finite values, then

(a). $\lim\limits_{x\to\infty}\left[f(x) \pm g(x)\right] = \lim\limits_{x\to\infty} f(x) \pm \lim\limits_{x\to\infty} g(x)$

(b). $\lim\limits_{x\to\infty} f(x)g(x) = \lim\limits_{x\to\infty} f(x) \cdot \lim\limits_{x\to\infty} g(x)$

(c). $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to\infty} f(x)}{\lim\limits_{x\to\infty} g(x)}$ if $\lim\limits_{x\to\infty} g(x) \neq 0$.

(d). For any constant *k*, $\lim\limits_{x\to\infty}\left[kf(x)\right] = k \lim\limits_{x\to\infty} f(x)$

(e). For any positive integer *n*,

(I). $\lim\limits_{x\to\infty}\left[f(x)\right]^n = \left[\lim\limits_{x\to\infty} f(x)\right]^n$.

(II). $\lim\limits_{x\to\infty} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to\infty} f(x)}$.

N.B. $\lim_{x\to\infty}\frac{1}{x}=0$

Example 1 Evaluate

(a) $\lim_{x\to\infty}\frac{3x^2-2x+7}{5x^2+x-1}$ **(b).** $\lim_{x\to\infty}\left(\sqrt{x^4+1}-x^2\right)$

Theorem Let $f(x)$ and $g(x)$ be two functions defined on *R*. Suppose X is a positive real number.

(a). If $f(x)$ is bounded for $x>X$ and $\lim_{x\to\infty}g(x)=0$ then $\lim_{x\to\infty}f(x)g(x)=0$.

(b). If $f(x)$ is bounded for $x>X$ and $\lim_{x\to\infty}g(x)=\pm\infty$, then $\lim_{x\to\infty}[f(x)\pm g(x)]=\pm\infty$.

(c). If $\lim_{x\to\infty}f(x)\neq 0$ and $f(x)$ is non-zero for $x>X$, and $\lim_{x\to\infty}g(x)=\pm\infty$, then $\lim_{x\to\infty}f(x)g(x)=\pm\infty$.

Example 2 Evaluate

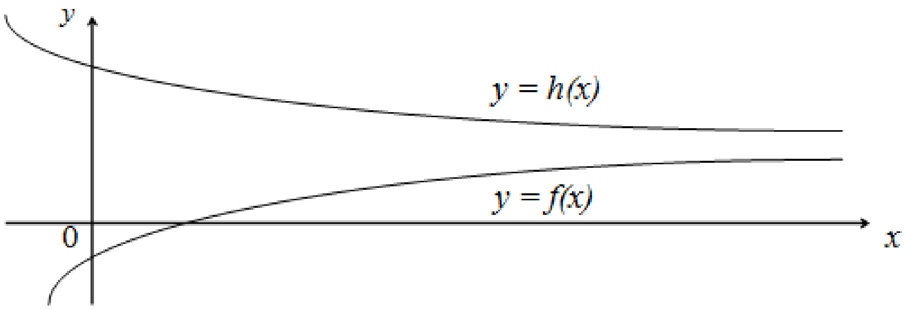
(a). $\lim_{x\to\infty}\frac{\sin x}{x}$. **(b)**. $\lim_{x\to\infty}e^{-x}\cos x$. **(c).** $\lim_{x\to\infty}\left(e^x\pm\sin x\right)$.

(d). $\lim_{x\to\infty}\frac{ex^e}{1+x}$. **(e).** $\lim_{x\to\infty}\frac{x+\cos x}{x+1}$.

Theorem SANDWICH THEOREM FOR FUNCTIONS

Let $f(x)$, $g(x)$, $h(x)$ be three functions defined on *R*.

If $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}h(x)=a$ and there exists a positive real number X such that when $x>X$, $f(x)\leq g(x)\leq h(x)$, then $\lim_{x\to\infty}g(x)=a$.



Example 3

(a). Show that for $x>0$, $\frac{(x+1)(x+3)}{x+2}<\sqrt{(x+1)(x+3)}<x+2$.

(b). Hence find $\lim_{x\to\infty}\left[\sqrt{(x+1)(x+3)}-x\right]$.

N.B. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$ for any positive integer *n*.

Example 4 Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ by sandwich rule.

Theorem $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \quad \Leftrightarrow \quad \lim_{y\to 0}(1+y)^{\frac{1}{y}} = e$.

Example 5 Evaluate

(a). $\lim_{x\to 0}(1-x)^{\frac{1}{x}}$. **(b).** $\lim_{x\to\infty}\left(\frac{x^2+1}{x^2-1}\right)^{x^2}$. **(c).** $\lim_{x\to 1} x^{\frac{1}{1-x}}$.

N.B. 1. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$. 2. $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$.

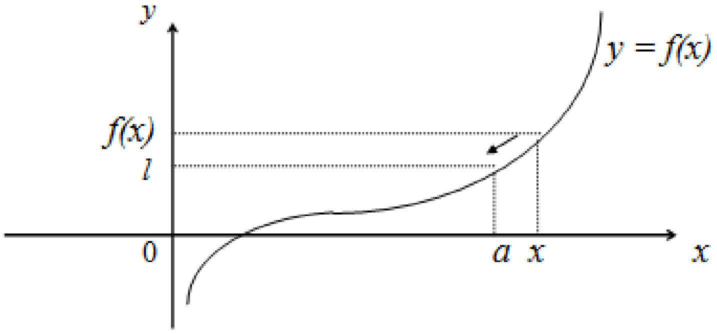
Exercise

(a). $\lim_{x\to 0}(1-3x)^{\frac{1}{x}}$. **(b).** $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{-x}$.

(c). $\lim_{x\to\infty}(1+\tan x)^{\cot x}$. **(d).** $\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x$.

B. LIMIT OF A FUNCTION AT A POINT

Definition Let $f(x)$ be a function defined on R. $\lim_{x\to a} f(x) = l$ means that for any $\varepsilon > 0$, there exists $\delta > 0$ such that when $0 < |x-a| < \delta$, $|f(x)-l| < \varepsilon$.



N.B.

(1). $\lim_{x\to a} f(x) = l$ means that the difference between $f(x)$ and l can be made arbitrarily small when x is sufficiently close to a.

(2). If $f(x)$ is a polynomial, then $\lim_{x\to a} f(x) = f(a)$.

(3). In general $\lim_{x\to a} f(x) \neq f(a)$.

(4). $f(x)$ may not defined $x = a$ even through $\lim_{x\to a} f(x)$ exists.

Example 6 $\lim_{x\to 3}\left(x^2-2x+5\right) = 3^2 - 2(3) + 5 = 8$, $\lim_{x\to\pi}\cos x = \cos\pi = -1$.

Example 7 Let $f(x)=\begin{cases}3 \text{ if } x\neq 5\\ 1 \text{ if } x=5\end{cases}$, then $\lim\limits_{x\to 5} f(x)=3$ but $f(5)=1$.

Example 8 Consider the function $f(x)=\begin{cases}2x+1, x\neq 2\\ 4, x=2\end{cases}$, $f(x)$ is discontinuous at $x=2$ since $\lim\limits_{x\to 2} f(x)=\lim\limits_{x\to 2}(2x+1)=5\neq f(2)$.

Example 9 Let $f(x)=\dfrac{x^2-4}{x-2}$, then $\lim\limits_{x\to 2} f(x)=\lim\limits_{x\to 2}\dfrac{(x-2)(x+2)}{x-2}=\lim\limits_{x\to 2}(x+2)=4$.

But $f(2)$ is not defined on *R*.

N.B. $\lim\limits_{x\to a} f(x)=l$ is equivalent to:

(I). $\lim\limits_{x\to a}\left[f(x)-l\right]=0$;

(II). $\lim\limits_{x\to a}\left|f(x)-l\right|=0$;

(III). $\lim\limits_{x\to a^-} f(x)=l=\lim\limits_{x\to a^+} f(x)$;

(IV). $\lim\limits_{h\to 0} f(a+h)=l$.

Theorem Rules of Operations on Limits

Let $f(x)$ and $g(x)$ be two functions defined on an interval containing a, possibly except *a*.

If $\lim\limits_{x\to a} f(x)=h$ and $\lim\limits_{x\to a} g(x)=k$, then

(a). $\lim\limits_{x\to a}\left[f(x)\pm g(x)\right]=h\pm k$.

(b). $\lim\limits_{x\to a}\left[f(x)g(x)\right]=hk$.

(c). $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{h}{k}$, (if $k\neq 0$)

Example 10 Evaluate

(a). $\lim\limits_{x\to -1}\dfrac{x^3+1}{x+1}$. **(b).** $\lim\limits_{x\to 0}\dfrac{(1+x)^n-1}{x}$.

(c). $\lim\limits_{x\to 2}\dfrac{2-x}{3-\sqrt{x^2+5}}$. **(d).** $\lim\limits_{x\to +\infty}\left(\sqrt{x^2+x}-x\right)$.

Example 11 Evaluate

(a). $\lim\limits_{x\to 3}\dfrac{1-x}{2-2x^2}$. **(b).** $\lim\limits_{x\to 1}\dfrac{1-x}{2-2x^2}$.

(c). $\lim\limits_{x\to 2}\dfrac{x-2}{\sqrt{x^2-2}-\sqrt{2}}$. **(d).** $\lim\limits_{x\to 1}\dfrac{1-\sqrt{x}}{1-\sqrt[3]{x}}$.

Example 12 **(a)**. Prove that for $x \in \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$, $2 < \frac{\sin x \tan x}{1-\cos x} < \frac{2}{\cos x}$.

(b). Hence, deduce that $\lim_{x \to 0} \frac{\sin x \tan x}{1-\cos x} = 2$.

3. Properties of Limit of a Function

Theorem Uniqueness of Limit Value

If $\lim_{x \to a} f(x) = h$ and $\lim_{x \to a} f(x) = k$, then $h = k$.

Theorem Heine Theorem

$\lim_{x \to a} f(x) = l \iff \lim_{n \to \infty} f(x_n) = l$ where $x_n \to a$ as $n \to \infty$.

Example 1 Prove that $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

Example 2 If $\lim_{x \to -1} \frac{x^3 - ax^2 - x + 4}{x+1}$ exists, find the value of a and the limit value.

4. Two Important Limits

Theorem $\lim_{x \to 0} \frac{\sin x}{x} = 1$

N.B. **(1).** $\lim_{x \to \infty} \frac{\sin x}{x} = \lim_{x \to \infty} \frac{\cos x}{x} =$. **(2).** $\lim_{x \to 0} \frac{x}{\sin x} = \lim_{x \to 0} \frac{\tan x}{x} =$.

(3). $\lim_{x \to 0} \frac{\cos x}{x} =$. **(4).** $\lim_{x \to 0} \sin \frac{1}{x} =$.

Example 1 Find the limits of the following functions:

(a). $\lim_{x \to 0} \frac{\tan x}{x}$. **(b).** $\lim_{x \to 0} \frac{1-\cos x}{x^2}$.

(c). $\lim_{x \to \frac{\pi}{2}} \left(x - \frac{\pi}{2}\right) \sec x$. **(d).** $\lim_{x \to 0} \frac{\sin 4x}{\sin 5x}$.

Example 2 Find the limits of the following functions:

(a). $\lim_{x \to 1} \frac{\sin(x-1)}{x^2 - 3x + 2}$. **(b).** $\lim_{x \to 0} \frac{\cos ax - \cos bx}{x^2}$.

(c). $\lim_{x \to 0} \frac{3\sin \pi x - \sin 3\pi x}{x^3}$.

Example 3 Let θ be a real number such that $0 < \theta < \frac{\pi}{2}$

By using the formula $\sin\frac{\theta}{2^k} = 2\sin\frac{\theta}{2^{k+1}}\cos\frac{\theta}{2^{k+1}}$ show that

$$\sin\theta = 2^{n-1}\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}...\cos\frac{\theta}{2^{n-1}}\sin\frac{\theta}{2^{n-1}}$$

Hence, find the limit value $\lim_{n\to\infty}\left(\cos\frac{x}{2}\cos\frac{x}{2^2}...\cos\frac{x}{2^n}\right)$

5. Left and Right Hand Limits

Theorem $\lim_{x\to a} f(x) = l$ if $\lim_{x\to a^+} f(x) = l$ then $x > a$, $\lim_{x\to a^-} f(x) = l$ then $x < a$, $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = l$.

Example 1 Show that each of the following limits does not exist.

(a). $\lim_{x\to 2}\sqrt{x-2}$. **(b).** $\lim_{x\to 0}\frac{|x|}{x}$. **(c).** $\lim_{x\to 0} e^{\frac{1}{x}}$.

Example 2 By Sandwich rule, show that $\lim_{x\to 0}\left(a^{\frac{1}{x}} + b^{\frac{1}{x}}\right)^x$ does not exist for $a > b > 0$

Solution If $a > b > 0$ then $\frac{b}{a} < 1$

If $a > b > 0$ and $x < 0$ then

If $x > 0$ then $\left(\frac{b}{a}\right)^{\frac{1}{x}} < 1^{\frac{1}{x}} = 1$

$$\left(\frac{b}{a}\right)^{\frac{1}{x}} > 1 \Leftrightarrow \left(\frac{b}{a}\right)^{\frac{1}{x}} < 1$$

$$a < \left(a^{\frac{1}{x}} + b^{\frac{1}{x}}\right)^x = a\left\{1 + \left(\frac{b}{a}\right)^{\frac{1}{x}}\right\}^x < a\cdot 2^x$$

$$b < \left(a^{\frac{1}{x}} + b^{\frac{1}{x}}\right)^x = b\left\{\left(\frac{a}{b}\right)^{\frac{1}{x}} + 1\right\}^x < b\cdot 2^x$$

As $\lim_{x\to 0^+} 2^x = 1$, by sandwich rule, $\lim_{x\to 0}\left(a^{\frac{1}{x}} + b^{\frac{1}{x}}\right)^x = a$.

As $\lim_{x\to 0^-} b\cdot 2^2 = b$, by sandwich rule, we have $\lim_{x\to 0}\left(a^{\frac{1}{x}} + b^{\frac{1}{x}}\right)^x = b$.

Since $a \neq b$, $\lim_{x \to 0}\left(a^{\frac{1}{x}} + b^{\frac{1}{x}}\right)^x$ does not exist. (Why?).

6. Continuous Functions

N.B. In general $\lim_{x \to a} f(x) \neq f(a)$.

Definition Let $f(x)$ be a function defined on R. $f(x)$ is said to be continuous at $x = a$ if and only if $\lim_{x \to a} f(x) = f(a)$.

N.B. This is equivalent to the definition:

A function $f(x)$ is continuous at $x = a$, if and only if

(a). $f(x)$ is well-defined at $x = a$, i.e. $f(a)$ exists and $f(a)$ is a finite value,

(b). $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = f(a)$.

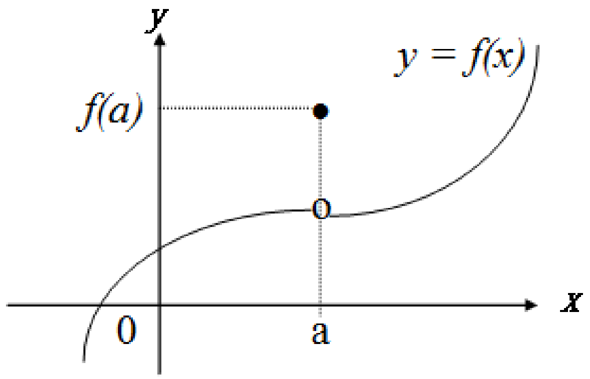
Remark Sometimes, the second condition may be written as $\lim_{h \to 0} f(a+h) = f(a)$.

Example 1 Show that the function $f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuous at $x = 0$.

Definition A function is discontinuous at $x = a$ if it is not continuous at that point a.

There are four kinds of discontinuity:

(1) Removable discontinuity:



$\lim_{x \to a} f(x)$ exists but not equal to $f(a)$.

Example 2 Show that $f(x) = \begin{cases} 2x+1, & x \neq 2 \\ 4, & x = 2 \end{cases}$ is discontinuous at $x = 2$.

Solution Since $f(2) = 4$ and $\lim_{x \to 2} f(x) = \lim_{x \to 2}(2x+1) = 5 \neq f(2)$,

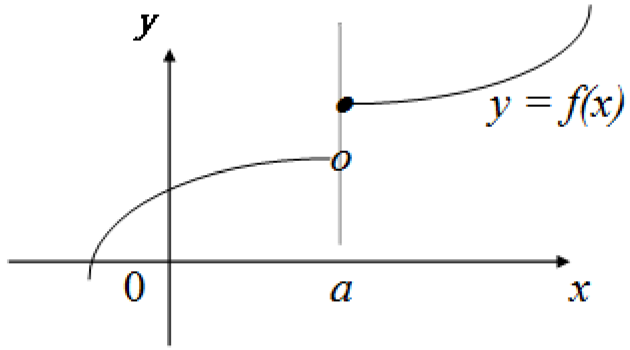
so $f(x)$ is discontinuous at $x = 2$.

Example 3 Let $f(x)=\begin{cases}\dfrac{1-\cos x}{x^2}, & x\neq 0\\ a, & x=0\end{cases}$.

(a). Find $\lim_{x\to 0} f(x)$.

(b). Find *a* if $f(x)$ is continuous at $x=0$.

(2) Jump discontinuity:



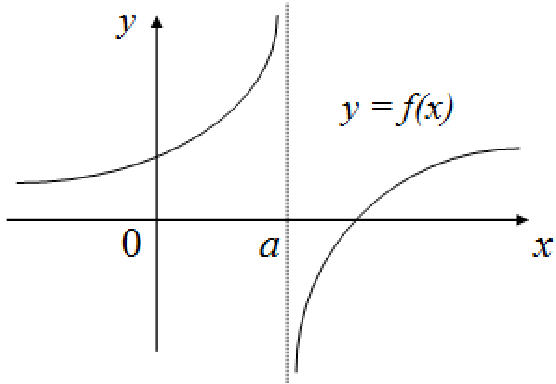
$\lim_{x\to a^-} f(x)\neq \lim_{x\to a^+} f(x)$

Example 4 Show that the $f(x)=[x]$ is discontinuous at 2.

Example 5 Find the points of discontinuity of the function $g(x)=x-[x]$.

Example 6 Let $f(x)=\begin{cases}x^3-2x+1, & x\geq 0\\ \sin x, & x<0\end{cases}$. Show that $f(x)$ is discontinuous at $x=0$

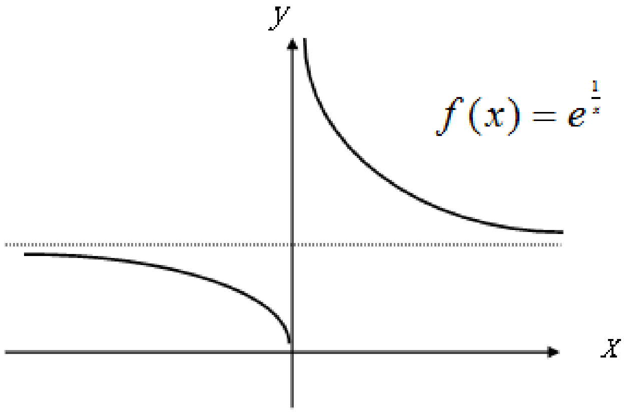
(3) Infinite discontinuity:



$\lim_{x\to a} f(x)=\infty$ or $\lim_{x\to a} f(x)=-\infty$ i.e. limit does not exist.

Example 7 Show that the function $f(x)=\dfrac{1}{x}$ is discontinuous at $x=0$

(4) At least one of the one-side limit does not exist.



$\lim_{x\to a^+} f(x)=\infty$ or $\lim_{x\to a^-} f(x)=-\infty$ do not exist.

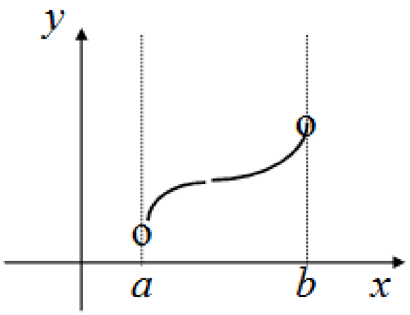
Example 8 Since $\lim_{x\to 0^-} e^{\frac{1}{x}}=0$ and $\lim_{x\to 0^+} e^{\frac{1}{x}}=\infty$, so the function $f(x)=e^{\frac{1}{x}}$ is discontinuous at $x=0$.

Definition

(1). A function $f(x)$ is called a continuous function in an open interval (a,b) if it is continuous at every point in (a,b).

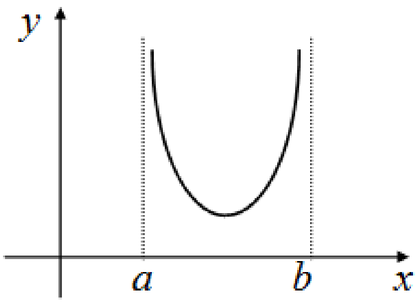
(2). A function $f(x)$ is called a continuous function in an closed interval $[a,b]$ if it is continuous at every point in (a,b) and $\lim_{x\to a^+} f(x)=f(a)$ and $\lim_{x\to b^-} f(x)=f(b)$.

(1)

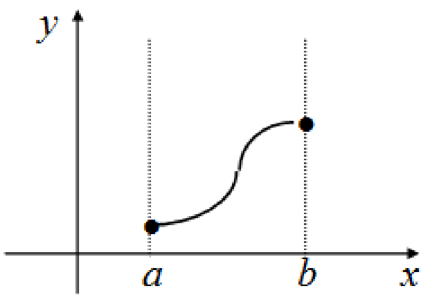


$f(x)$ is continuous on (a,b)

(2)



$f(x)$ is continuous on (a,b)



$f(x)$ is continuous on $[a,b]$

N.B.

(1). $f(x)$ is continuous at $x=a \Leftrightarrow \lim_{x\to a} f(x)=f(a) \Leftrightarrow \lim_{h\to 0} f(a+h)=f(a)$.

(2). $f(x)$ is continuous at every x on $R \Leftrightarrow \lim_{h\to 0} f(x+h)=f(x)$ for all $x\in R$.

Example 9 Let $f(x)$ be a real-value function such that.

(1). $f(x+y)=f(x)f(y)$ for all real numbers x and y,

(2). $f(x)$ is continuous at $x=0$ and $f(0)=1$.

Show that $f(x)$ is continuous at every $x\in R$.

7. Properties of Continuous Functions

A Continuity of Elementary Functions

Theorem Rules of Operations on Continuous Functions

If $f(x)$ and $g(x)$ are two functions continuous at $x=a$, then so are $f(x)\pm g(x)$, $f(x)g(x)$ and $\dfrac{f(x)}{g(x)}$ provided $g(a)\neq 0$.

Example 1 Let $h(x)=\dfrac{1+\cos x}{x^2}$. Since $f(x)=1+\cos x$ and $g(x)=x^2$ are two functions continuous everywhere, but $g(0)=0$, so h is continuous everywhere except $x=0$.

Example 2 Let $h(x)=\dfrac{1+\cos x}{x^2+1}$. Since $f(x)=1+\cos x$ and $g(x)=x^2+1\ (\neq 0)$ are two functions continuous everywhere, so h is continuous everywhere.

Theorem Let $g(x)$ be continuous at $x=a$ and $f(x)$ be continuous at $x=g(a)$, then $f_0\ g$ is continuous at $x=a$.

Example 3 $\sin e^x$, $e^{\sin x}$ are continuous functions.

Example 4 Show that $f(x)=\cos x^2$ is continuous at every $x\in R$.

Example 5 Let f and g be two functions defined as $f(x)=\dfrac{x+|x|}{2}$ for all x and

$$g(x)=\begin{cases} x \text{ if } x<0 \\ x^2 \text{ if } x\geq 0 \end{cases}.$$

For what values of x is $f\left[g(x)\right]$ continuous?

Theorem Let $\lim_{x\to\infty} f(x) = A$ and $g(x)$ be an elementary function (such as $\sin x$, $\cos x$, $\sin^{-1} x$, e^x, $\ln x$, … , etc.) for which $g(A)$ is defined, then

$$\lim_{x\to\infty} g[f(x)] = g\left[\lim_{x\to\infty} f(x)\right] = g(A)$$

Example 6 $\lim_{x\to 1}\cos(\tan \pi x) = \cos\left(\lim_{x\to 1}\tan \pi x\right) = 0$

Example 7 Evaluate $\lim_{x\to\infty}\frac{\ln(1+x)}{x}$.

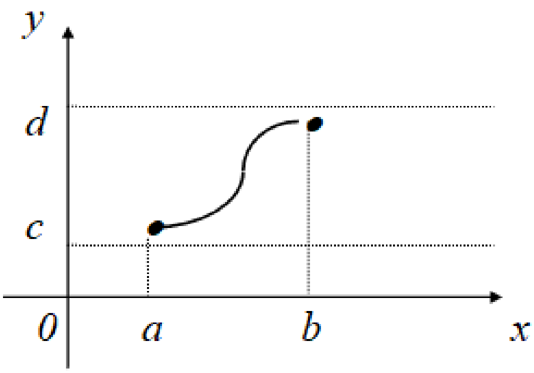
Example 8 Evaluate $\lim_{x\to 0}\tan\left(\frac{\sin \pi x}{x}\right)$.

B. Properties of Continuous Function

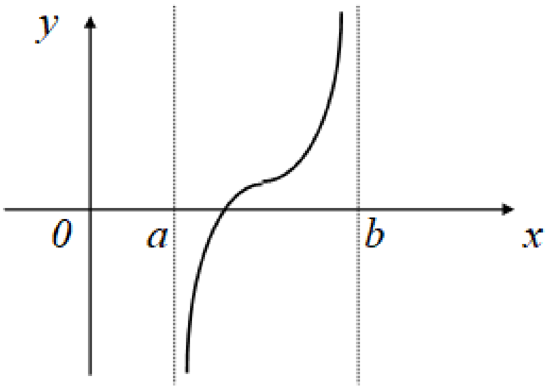
(P1) If $f(x)$ is continuous on $[a,b]$, then $f(x)$ is bounded on $[a,b]$.

(P2) But $f(x)$ is continuous on (a,b) cannot imples that $f(x)$ is bounded on (a,b).

(1) $c < f(x) < d$



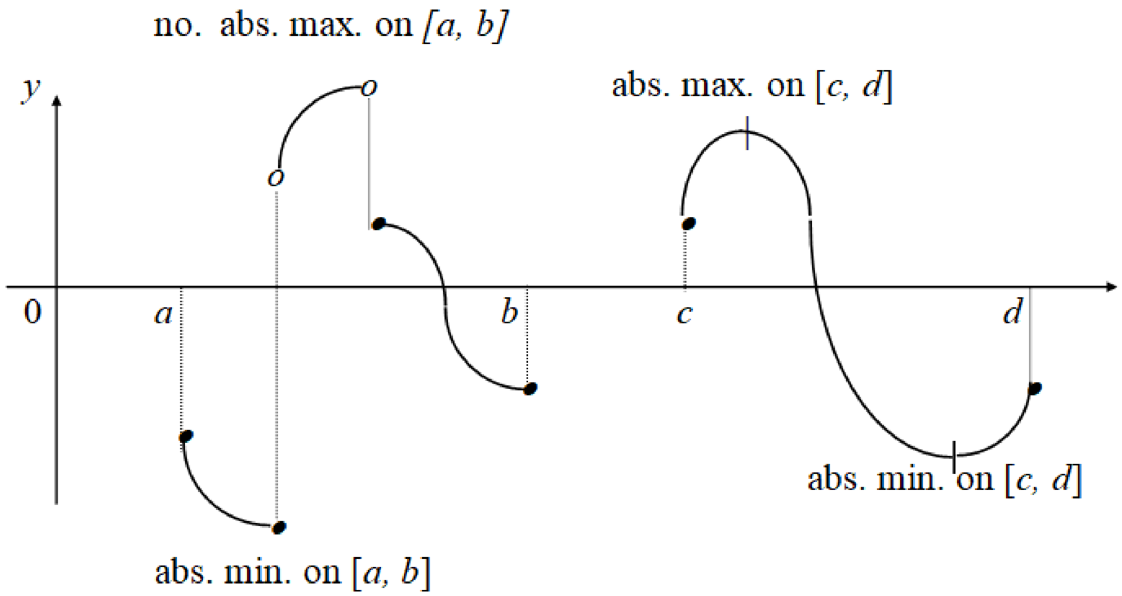
(2) $f(x)$ is not bounded on (a,b)



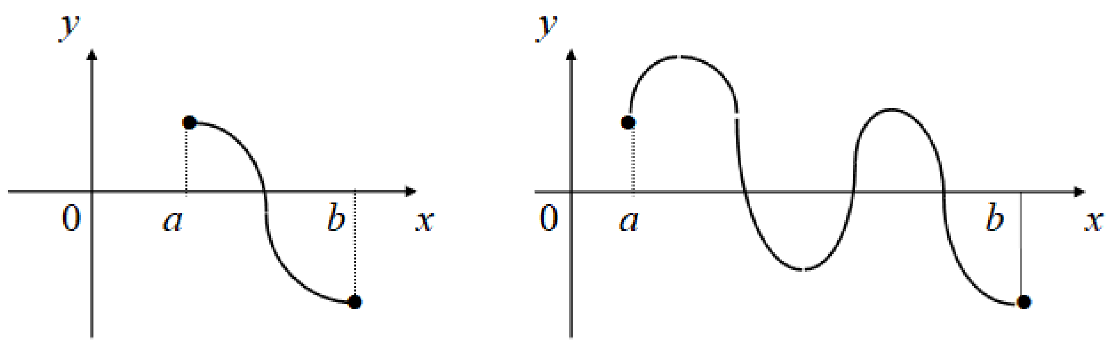
Example 9 $f(x) = \cot x$ continuous on $(0,\pi)$, but it is not bounded on $(0,\pi)$.

(P3) If $f(x)$ is continuous on $[a,b]$, then it will attain an absolute maximum and absolute minimum on $[a,b]$.

i.e. If $f(x)$ is continuous on $[a,b]$, there exist $x_1 \in [a,b]$ such that $f(x_1) \geq f(x)$ and $x_2 \in [a,b]$ such that $f(x_2) \leq f(x)$ for all $x \in [a,b]$. x_1 is called the absolute maximum of the function and x_2 is called the absolute minimum of the function.



(P4) If $f(x)$ is continuous on $[a,b]$ and $f(a)f(b) < 0$, then there exists $c \in [a,b]$ such that $f(c) = 0$.



Example 10 Let $f(x) = \cos x - 0.6$ which is continuous on $\left[0, \frac{\pi}{2}\right]$ and $f(0) = 0.4$ and $f\left(\frac{\pi}{2}\right) = -0.6$. Hence, there exists a real number $c \in \left[0, \frac{\pi}{2}\right]$ such that $f(c) = 0$.

Example 11 Prove that the equation $x \cdot e^x = 2$ has at least one real root in $[0,1]$.

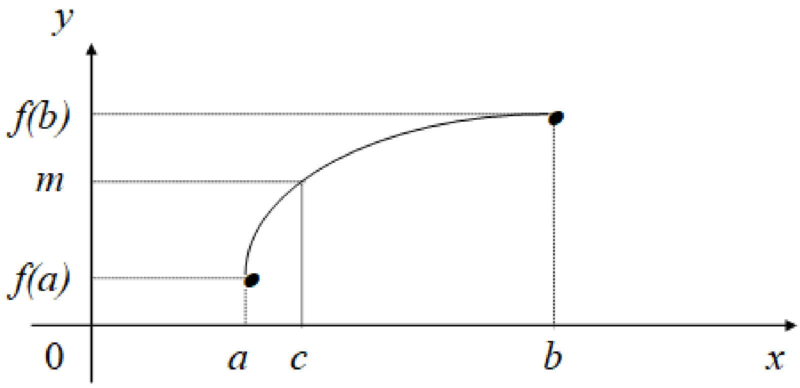
Example 12 Le $f(x) = x^3 + 2x$.

(a). Show that $f(x)$ is a strictly increasing function.

(b). Hence, show that the equation $f(x) = 0$ has a unique real root.

(P5) Intermediate Value Theorem

If $f(x)$ is continuous on $[a,b]$, then for any real number m lying between $f(a)$ and $f(b)$, there corresponds a number $c \in [a,b]$ such that $f(c)=m$.



(P6) Let $f(a)=c$ and $f(b)=d$, if f is continuous and strictly increasing on $[a,b]$, then f^{-1} is also continuous and strictly increasing on $[c,d]$ (or $[d,c]$).

Example 13 $f(x)=x^2$ is strictly increasing and continuous on $[0,5]$, so $f^{-1}(x)=\sqrt{x}$ is also strictly increasing and continuous on $[0,25]$.

Example 14 $f(x)=\cos x$ is strictly decreasing and continuous on $[0,\pi]$, so $f^{-1}(x)=\cos^{-1}x$ is also strictly decreasing and continuous on $[-1,1]$.

Example 15

(a). Suppose that the function satisfies $f(x+y)=f(x)+f(y)$ for all real *x* and *y* and $f(x)$ is continuous at $x=0$. Show that $f(x)$ is continuous at all *x*.

(b). A function $f(x)=\begin{cases} \dfrac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x} & \text{if } x \neq 0 \\ 0, & \text{if } x=0 \end{cases}$.

Is $f(x)$ continuous at $x=0$? If not, how can we redefine the value of $f(x)$ at $x=0$ so that it is continuous at $x=0$?

Example 16 Let $f(x)$ be a continuous function defined for $x>0$ and for any *x* $y>0$, $f(xy)=f(x)+f(y)$

(a). Find $f(1)$.

(b). Let a be a positive real number. Prove that $f(a^r)=r\cdot f(a)$ for any non-negative rational number.

(c). It is known that for all real numbers *x*, there exists a sequence $\{x_n\}$ of rational numbers such that $\lim\limits_{x\to\infty} x_n = x$.

(I). Show that $f(a^x)=x\cdot f(a)$ for all $x>0$, where a is a positive real constant.

(II). Hence show that $f(x) = c \ln x$ for all $x > 0$, where c is a constant.

Example 17 Let f be a real-valued continuous function defined on the set R such that $f(x+y) = f(x) + f(y)$ for all $x,\ y \in R$.

(a). Show that

(I). $f(0) = 0$.

(II). $f(-x) = -f(x)$ for any $x \in R$.

(b). Prove that $f(nx) = nf(x)$ for all integers *n*.

Hence show that $f(r) = rf(1)$ for any rational number *r*.

(c). It is known that for all $x \in R$, there exists a sequence $\{a_n\}$ of rational numbers such that $\lim_{n\to\infty} a_n = x$.

Using **(b),** prove that there exists a constant k such that $f(x) = kx$ for all $x \in R$.

Example 18 Let R denote the set of all real numbers and *f*: $R \to R$ a continuous function not identically zero such that $f(x+y) = f(x)f(y)$ for all $x\ \ y \in R$,

(a). Show that

(I). $f(x) \neq 0$ for any $x \in R$.

(II). $f(x) > 0$ for any $x \in R$.

(III). $f(0) = 1$.

(IV). $f(-x) = \left[f(x)\right]^{-1}$.

(b). Prove that for any rational number *r*, $f(rx) = \left[f(x)\right]^r$.

Hence prove that there exists a constant a such that $f(x) = a^x$ for all $x \in R$.

Derivatives

8. The Definition of the Derivative

In this section we saw that the computation of the slope of a tangent line, the instantaneous rate of change of a function, and the instantaneous velocity of an object at $x = a$ all required us to compute the following limit.

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

We also saw that with a small change of notation this limit could also be written as,

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

This is such an important limit and it arises in so many places that we give it a name. We call it a derivative. Here is the official definition of the derivative.

Definition of the Derivative

The derivative of $f(x)$ with respect to *x* is the function $f'(x)$ and is defined as,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Note that we replaced all the $a's$ in with $x's$ to acknowledge the fact that the derivative is really a function as well. We often «read» $f'(x)$ as «*f* prime of *x*».

Let's compute a couple of derivatives using the definition.

Example 1. Find the derivative of the following function using the definition of the derivative.

$$f(x)=2x^2-16x+35$$

Solution

So, all we really need to do is to plug this function into the definition of the derivative, and do some algebra. While, admittedly, the algebra will get somewhat unpleasant at times, but it's just algebra so don't get excited about the fact that we're now computing derivatives.

First plug the function into the definition of the derivative.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{2(x+h)^2-16(x+h)+35-(2x^2-16x+35)}{h}$$

Be careful and make sure that you properly deal with parenthesis when doing the subtracting.

Now, we know from the previous section that we can't just plug in $h=0$ since this will give us a division by zero error. So, we are going to have to do some work. In this case that means multiplying everything out and distributing the minus sign through on the second term. Doing this gives,

$$f'(x)=\lim_{h\to 0}\frac{2x^2+4xh+2h^2-16x-16h+35-2x^2+16x-35}{h}=$$

$$=\lim_{h\to 0}\frac{4xh+2h^2-16h}{h}$$

Notice that every term in the numerator that didn't have an *h* in it canceled out and we can now factor an *h* out of the numerator which will cancel against the *h* in the denominator. After that we can compute the limit.

$$f'(x)=\lim_{h\to 0}\frac{h(4x+2h-16)}{h}=\lim_{h\to 0}4x+2h-16=4x-16$$

So, the derivative is,

$$f'(x)=4x-16$$

Example 2. Find the derivative of the following function using the definition of the derivative.

$$g(t)=\frac{t}{t+1}$$

Solution

This one is going to be a little messier as far as the algebra goes. However, outside of that it will work in exactly the same manner as the previous examples. First, we plug the function into the definition of the derivative,

$$g'(t) = \lim_{h\to 0}\frac{g(t+h)-g(t)}{h} = \lim_{h\to 0}\frac{1}{h}\left(\frac{t+h}{t+h+1}-\frac{t}{t+1}\right)$$

Note that we changed all the letters in the definition to match up with the given function. Also note that we wrote the fraction a much more compact manner to help us with the work.

As with the first problem we can't just plug in $h=0$. So, we will need to simplify things a little. In this case we will need to combine the two terms in the numerator into a single rational expression as follows.

$$g'(t) = \lim_{h\to 0}\frac{1}{h}\left(\frac{(t+h)(t+1)-t(t+h+1)}{(t+h+1)(t+1)}\right) =$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{t^2+t+th+h-(t^2+th+t)}{(t+h+1)(t+1)}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{h}{(f+h+1)(t+1)}\right)$$

Before finishing this let's note a couple of things. First, we didn't multiply out the denominator. Multiplying out the denominator will just overly complicate things so let's keep it simple. Next, as with the first example, after the simplification we only have terms with $h's$ in them left in the numerator and so we can now cancel an *h* out.

So, upon canceling the *h* we can evaluate the limit and get the derivative.

$$g'(t) = \lim_{h\to 0}\frac{1}{(t+h+1)(t+1)} = \frac{1}{(t+1)(t+1)} = \frac{1}{(t+1)^2}$$

The derivative is then,

$$g'(t) = \frac{1}{(t+1)^2}$$

Example 3. Find the derivative of the following function using the definition of the derivative.

$$R(z) = \sqrt{5z-8}$$

Solution

First plug into the definition of the derivative as we've done with the previous two examples.

$$R'(z) = \lim_{h\to 0}\frac{R(z+h)-R(z)}{h} = \lim_{h\to 0}\frac{\sqrt{5(z+h)-8}-\sqrt{5z-8}}{h}$$

In this problem we're going to have to rationalize the numerator. You do remember rationalization from an Algebra class right? In an Algebra class you probably only rationalized the denominator, but you can also rationalize numerators. Remember that in rationalizing the numerator (in this case) we multiply both the numerator and denominator by the numerator except we change the sign between the two terms. Here's the rationalizing work for this problem,

$$R'(z) = \lim_{h\to 0} \frac{\left(\sqrt{5(z+h)-8} - \sqrt{5z-8})\right)}{h} \frac{\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}{\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)} =$$

$$= \lim_{h\to 0} \frac{5z+5h-8-(5z-8)}{h\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)} = \lim_{h\to 0} \frac{5h}{h\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}$$

Again, after the simplification we have only $h's$ left in the numerator. So, cancel the *h* and evaluate the limit.

$$R'(z) = \lim_{h\to 0} \frac{5}{\sqrt{5(z+h)-8} + \sqrt{5z-8}} =$$

$$= \frac{5}{\sqrt{5z-8} + \sqrt{5z-8}} = \frac{5}{2\sqrt{5z-8}}$$

And so we get a derivative of,

$$R'(z) = \frac{5}{2\sqrt{5z-8}}$$

Let's work one more example. This one will be a little different, but it's got a point that needs to be made.

Example 4. Determine $f'(0)$ for $f(x) = |x|$

Solution

Since this problem is asking for the derivative at a specific point we'll go ahead and use that in our work. It will make our life easier and that's always a good thing.

So, plug into the definition and simplify.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|0+h| - |0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

We saw a situation like this back when we were looking at limits at infinity. As in that section we can't just cancel the $h's$. We will have to look at the two one sided limits and recall that

$$|h| = \begin{cases} h & \text{if } h \ge 0 \\ -h & \text{if } h < 0 \end{cases}$$

$$\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-}(-1) = -1 \qquad \text{because } h<0 \text{ in a left-hand limit.}$$

$$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+}(1) = 1 \qquad \text{because } h>0 \text{ in a right-hand limit.}$$

The two one-sided limits are different and so

$$\lim_{h\to 0} \frac{|h|}{h}$$

doesn't exist. However, this is the limit that gives us the derivative that we're after.

If the limit doesn't exist then the derivative doesn't exist either.

In this example we have finally seen a function for which the derivative doesn't exist at a point. This is a fact of life that we've got to be aware of. Derivatives will not always exist. Note as well that this doesn't say anything about whether or not the derivative exists anywhere else. In fact, the derivative of the absolute value function exists at every point except the one we just looked at, $x=0$.

The preceding discussion leads to the following definition.

Definition

A function is called differentiable at $x=a$ if $f'(a)$ exists and $f(x)$ is called differentiable on an interval if the derivative exists for each point in that interval.

The next theorem shows us a very nice relationship between functions that are continuous and those that are differentiable.

Theorem

If $f(x)$ is differentiable at $x=a$ then $f(x)$ is continuous at $x=a$.

Note that this theorem does not work in reverse. Consider $f(x)=|x|$ and take a look at,

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}|x|=0=f(0)$$

So, is continuous at but we've just shown above in Example 4 that $f(x)=|x|$ is not differentiable at $x=0$.

Alternate Notation

Next, we need to discuss some alternate notation for the derivative. The typical derivative notation is the «prime» notation. However, there is another notation that is used on occasion so let's cover that.

Given a function $y=f(x)$ all of the following are equivalent and represent the derivative of $f(x)$ with respect to *x*.

$$f'(a)=y'=\frac{df}{dx}=\frac{dy}{dx}=\frac{d}{dx}\big(f(x)\big)=\frac{d}{dx}(y)$$

Because we also need to evaluate derivatives on occasion we also need a notation for evaluating derivatives when using the fractional notation. So, if we want to evaluate the derivative at $x=a$ all of the following are equivalent.

$$f'(a)=y'\big|_{x=a}=\left.\frac{df}{dx}\right|_{x=a}=\left.\frac{dy}{dx}\right|_{x=a}$$

As a final note in this section we'll acknowledge that computing most derivatives directly from the definition is a fairly complex (and sometimes painful) process filled with opportunities to make mistakes.

This does not mean however that it isn't important to know the definition of the derivative! It is an important definition that we should always know and keep in the back of our minds. It is just something that we're not going to be working with all that much.

9. Interpretation of the Derivative

Let's quickly reviewsome interpretations of the derivative. All of these interpretations arise from recalling how our definition of the derivative came about. The definition came about by noticing that all the problems that we worked in the previous sections required us to evaluate the same limit.

Rate of Change

The first interpretation of a derivative is rate of change. This was not the first problem that we looked , but it is the most important interpretation of the derivative. If $f(x)$ represents a quantity at any x then the derivative $f'(a)$ represents the instantaneous rate of change of $f(x)$ at $x=a$.

Example 1. Suppose that the amount of water in a holding tank at t minutes is given by $V(t)=2t^2-16t+35$. Determine each of the following.

(a) Is the volume of water in the tank increasing or decreasing at $t=1$ minute?
(b) Is the volume of water in the tank increasing or decreasing at $t=5$ minutes?
(c) Is the volume of water in the tank changing faster at $t=1$ or $t=5$ minutes?
(d) Is the volume of water in the tank ever not changing? If so, when?

Solution

In the solution to this example we will use both notations for the derivative just to get you familiar with the different notations.

We are going to need the rate of change of the volume to answer these questions. This means that we will need the derivative of this function since that will give us a formula for the rate of change at any time *t*. Now, notice that the function giving the volume of water in the tank is the same function that we saw in Example 1 in the last section except the letters have changed. The change in letters between the function in this example versus the function in the example from the last section won't affect the work and so we can just use the answer from that example with an appropriate change in letters.

The derivative is.

$$V'(t)=4t-16 \qquad \text{or} \qquad \frac{dV}{dt}=4t-16$$

Recall from our work in the first limits section that we determined that if the rate of change was positive then the quantity was increasing and if the rate of change was negative then the quantity was decreasing.

We can now work the problem.

(a) Is the volume of water in the tank increasing or decreasing at $t=1$ minute?

In this case all that we need is the rate of change of the volume at $t=1$ or,

$$V'(1) = -12 \quad \text{or} \quad \left.\frac{dV}{dt}\right|_{t-1} = -12$$

So, at $t = 1$ the rate of change is negative and so the volume must be decreasing at this time.

(b) Is the volume of water in the tank increasing or decreasing at $t = 5$ minutes?

Again, we will need the rate of change at $t = 5$.

$$V'(5) = 4 \quad \text{or} \quad \left.\frac{dV}{dt}\right|_{t-5} = 4$$

In this case the rate of change is positive and so the volume must be increasing at $t = 5$.

(c) Is the volume of water in the tank changing faster at $t = 1$ or $t = 5$ minutes?

To answer this question all that we look at is the size of the rate of change and we don't worry about the sign of the rate of change. All that we need to know here is that the larger the number the faster the rate of change. So, in this case the volume is changing faster at $t = 1$ than at $t = 5$.

(d) **Is the volume of water in the tank ever not changing? If so, when?**

The volume will not be changing if it has a rate of change of zero. In order to have a rate of change of zero this means that the derivative must be zero. So, to answer this question we will then need to solve

$$V'(t) = 0 \quad \text{or} \quad \frac{dV}{dt} = 0$$

This is easy enough to do.

$$4t - 16 = 0 \quad \Rightarrow \quad t = 4$$

So at $t = 4$ the volume isn't changing. Note that all this is saying is that for a brief instant the volume isn't changing. It doesn't say that at this point the volume will quit changing permanently.

If we go back to our answers from parts (a) and (b) we can get an idea about what is going on. At $t = 1$ the volume is decreasing and at $t = 5$ the volume is increasing. So, at some point in time the volume needs to switch from decreasing to increasing. That time is $t = 4$.

This is the time in which the volume goes from decreasing to increasing and so for the briefest instant in time the volume will quit changing as it changes from decreasing to increasing.

Note that one of the more common mistakes that students make in these kinds of problems is to try and determine increasing/decreasing from the function values rather than the derivatives. In this case if we took the function values at $t = 0$, $t = 1$ and $t = 5$ we would get,

$$V(0) = 35 \quad V(1) = 21 \quad V(5) = 5$$

Clearly as we go from $t = 0$ to $t = 1$ the volume has decreased. This might lead us to decide that AT $t = 1$ the volume is decreasing. However, we just can't say that. All we can say is that between $t = 0$ and $t = 1$ the volume has decreased at some point in time. The only way to know what is happening right at $t = 1$ is to compute $V'(1)$ and look at its sign to determine increasing/decreasing. In this case $V'(1)$ is negative and so the volume really is decreasing at $t = 1$.

Now, if we'd plugged into the function rather than the derivative we would have gotten the correct answer for $t = 1$ even though our reasoning would have been wrong. It's important to not let this give you the idea that this will always be the case. It just happened to work out in the case of $t = 1$.

To see that this won't always work let's now look at $t = 5$. If we plug $t = 1$ and $t = 5$ into the volume we can see that again as we go from $t = 1$ to $t = 5$ the volume has decreased. Again, however all this says is that the volume HAS decreased somewhere between $t = 1$ and $t = 5$. It does NOT say that the volume is decreasing at $t = 5$. The only way to know what is going on right at $t = 5$ is to compute $V'(5)$ and in this case $V'(5)$ is positive and so the volume is actually increasing at $t = 5$.

So, be careful. When asked to determine if a function is increasing or decreasing at a point make sure and look at the derivative. It is the only sure way to get the correct answer. We are not looking to determine is the function has increased/decreased by the time we reach a particular point. We are looking to determine if the function is increasing/decreasing at that point in question.

Slope of Tangent Line

This is the next major interpretation of the derivative. The slope of the tangent line to $f(x)$ at $x = a$ is $f'(a)$. The tangent line then is given by,

$$y = f(a) + f'(a)(x - a)$$

Example 2. Find the tangent line to the following function at $z = 3$.

$$R(z) = \sqrt{5z - 8}$$

Solution

We first need the derivative of the function and we found that in Example 3 in the last section. The derivative is,

$$R'(z) = \frac{5}{2\sqrt{5z - 8}}$$

Now all that we need is the function value and derivative (for the slope) at $z = 3$.

$$R(3) = \sqrt{7} \qquad m = R'(3) = \frac{5}{2\sqrt{7}}$$

The tangent line is then,

$$y = \sqrt{7} + \frac{5}{2\sqrt{7}}(z - 3)$$

Velocity

Recall that this can be thought of as a special case of the rate of change interpretation. If the position of an object is given by $f(t)$ after t units of time the velocity of the object at $t = a$ is given by $f'(a)$.

Example 3. Suppose that the position of an object after t hours is given by,

$$g(t) = \frac{t}{t+1}$$

Answer both of the following about this object.

(a) Is the object moving to the right or the left at $t = 10$ hours?

(b) Does the object ever stop moving?

Solution

Once again, we need the derivative and we found that in Example 2 in the last section. The derivative is,

$$g'(t) = \frac{1}{(t+1)^2}$$

(a) Is the object moving to the right or the left at $t = 10$ hours?

To determine if the object is moving to the right (velocity is positive) or left (velocity is negative) we need the derivative at $t = 10$.

$$g'(10) = \frac{1}{121}$$

So, the velocity at is positive and so the object is moving to the right at $t = 10$.

(b) Does the object ever stop moving?

The object will stop moving if the velocity is ever zero. However, note that the only way a rational expression will ever be zero is if the numerator is zero. Since the numerator of the derivative (and hence the speed) is a constant it can't be zero.

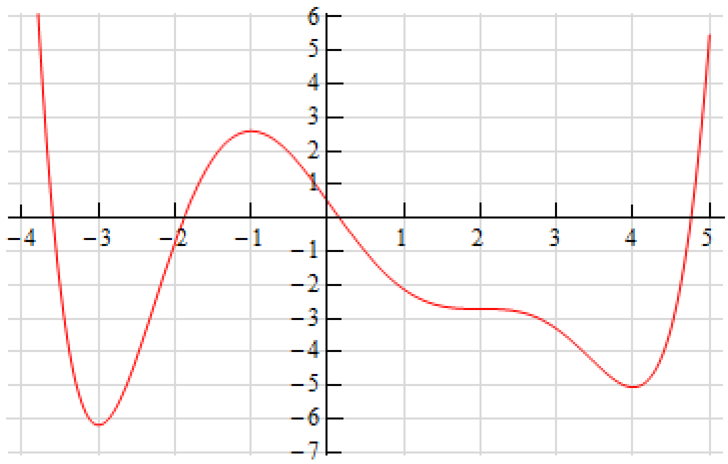
Therefore, the object will never stop moving.

In fact, we can say a little more here. The object will always be moving to the right since the velocity is always positive.

We've seen three major interpretations of the derivative here. You will need to remember these, especially the rate of change, as they will show up continually.

Before we finish this section let's work one more example that encompasses some of the ideas discussed here and is just a nice example to work.

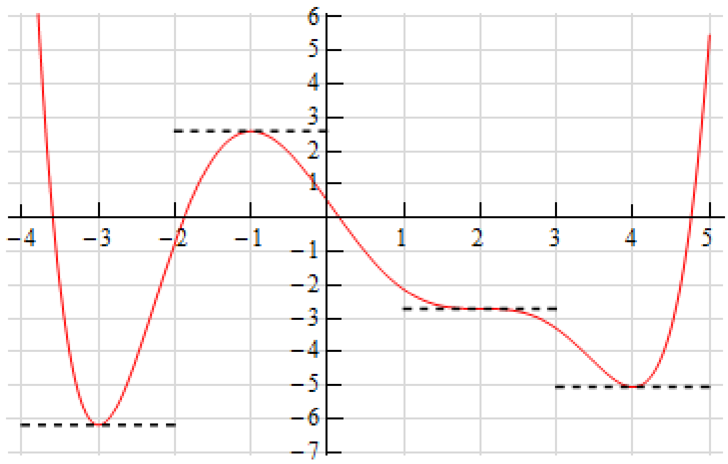
Example 4. Below is the sketch of a function $f(x)$. Sketch the graph of the derivative of this function, $f'(x)$.



Solution

At first glance this seems to an all but impossible task. However, if you have some basic knowledge of the interpretations of the derivative you can get a sketch of the derivative. It will not be a perfect sketch for the most part, but you should be able to get most of the basic features of the derivative in the sketch.

Let's start off with the following sketch of the function with a couple of additions.



Notice that at $x=-3$, $x=-1$, $x=2$ and $x=4$ the tangent line to the function is horizontal. This means that the slope of the tangent line must be zero. Now, we know that the slope of the tangent line at a particular point is also the value of the derivative of the function at that point. Therefore, we now know that,

$$f'(-3)=0 \qquad f'(-1)=0 \qquad f'(2)=0 \qquad f'(4)=0$$

This is a good starting point for us. It gives us a few points on the graph of the derivative. It also breaks the domain of the function up into regions where the function is increasing and decreasing. We know, from our discussions above, that if the function is increasing at a point then the derivative must be positive at that point. Likewise, we know that if the function is decreasing at a point then the derivative must be negative at that point.

We can now give the following information about the derivative.

$x < -3$	$f'(x) < 0$
$-3 < x < -1$	$f'(x) > 0$
$-1 < x < 2$	$f'(x) < 0$
$2 < x < 4$	$f'(x) < 0$
$x > 4$	$f'(x) > 0$

Remember that we are giving the signs of the derivatives here and these are solely a function of whether the function is increasing or decreasing. The sign of the function itself is completely immaterial here and will not in any way effect the sign of the derivative.

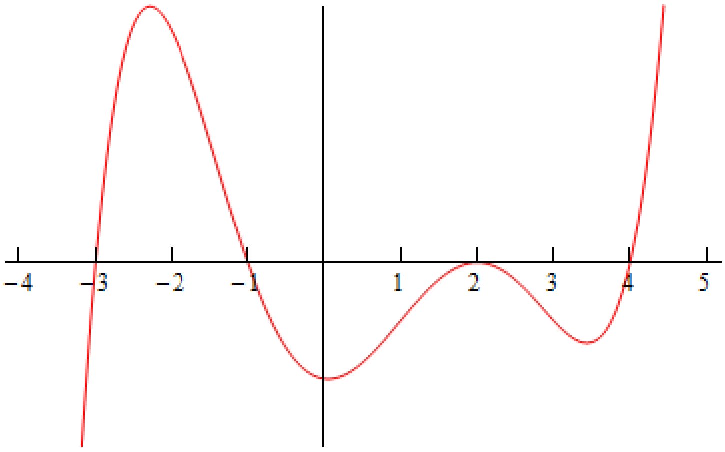
This may still seem like we don't have enough information to get a sketch, but we can get a little bit more information about the derivative from the graph of the function. In the range $x < -3$ we know that the derivative must be negative, however we can also see that the derivative needs to be increasing in this range. It is negative here until we reach $x = -3$ and at this point the derivative must be zero. The only way for the derivative to be negative to the left of $x = -3$ and zero at $x = -3$ is for the derivative to increase as we increase x towards $x = -3$.

Now, in the range $-3 < x < -1$ we know that the derivative must be zero at the endpoints and positive in between the two endpoints. Directly to the right of $x = -3$ the derivative must also be increasing (because it starts at zero and then goes positive — therefore it must be increasing). So, the derivative in this range must start out increasing and must eventually get back to zero at $x = -1$. So, at some point in this interval the derivative must start decreasing before it reaches $x = -1$. Now, we have to be careful here because this is just general behavior here at the two endpoints. We won't know where the derivative goes from increasing to decreasing and it may well change between increasing and decreasing several times before we reach $x = -1$. All we can really say is that immediately to the right of $x = -3$ the derivative will be increasing and immediately to the left of $x = -1$ the derivative will be decreasing.

Next, for the ranges $-1 < x < 2$ and $2 < x < 4$ we know the derivative will be zero at the endpoints and negative in between. Also, following the type of reasoning given above we can see in each of these ranges that the derivative will be decreasing just to the right of the left-hand endpoint and increasing just to the left of the right hand endpoint.

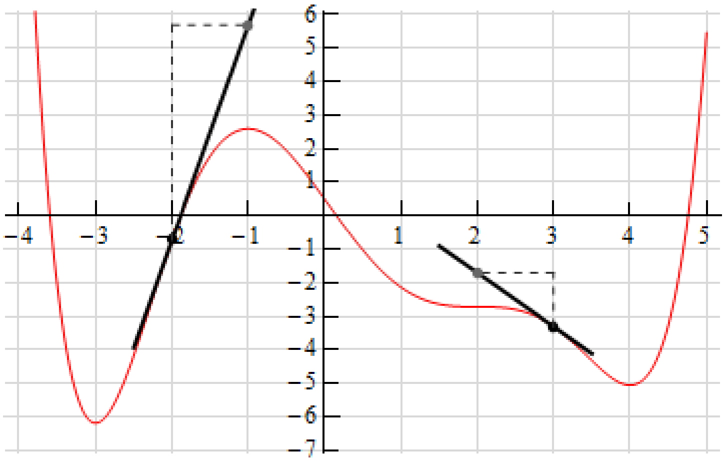
Finally, in the last region $x > 4$ we know that the derivative is zero at $x = 4$ and positive to the right of $x = 4$. Once again, following the reasoning above, the derivative must also be increasing in this range.

Putting all of this material together (and always taking the simplest choices for increasing and/or decreasing information) gives us the following sketch for the derivative.



Note that this was done with the actual derivative and so is in fact accurate. Any sketch you do will probably not look quite the same. The «humps» in each of the regions may be at different places and/or different heights for example. Also, note that we left off the vertical scale because given the information that we've got at this point there was no real way to know this information.

That doesn't mean however that we can't get some ideas of specific points on the derivative other than where we know the derivative to be zero. To see this let's check out the following graph of the function (not the derivative, but the function).

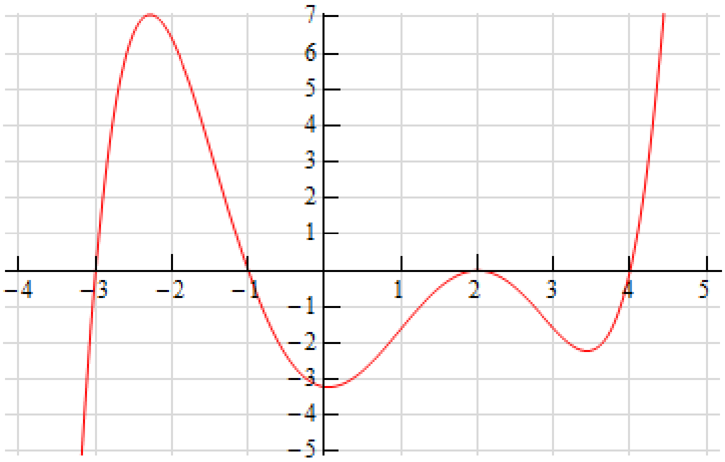


At $x=-2$ and $x=3$ we've sketched in a couple of tangent lines. We can use the basic rise/run slope concept to estimate the value of the derivative at these points.

Let's start at $x=3$. We've got two points on the line here. We can see that each seem to be about one-quarter of the way off the grid line. So, taking that into account and the fact that we go through one complete grid we can see that the slope of the tangent line, and hence the derivative, is approximately –1.5.

At $x=-2$ it looks like (with some heavy estimation) that the second point is about 6.5 grids above the first point and so the slope of the tangent line here, and hence the derivative, is approximately 6.5.

Here is the sketch of the derivative with the vertical scale included and from this we can see that in fact our estimates are pretty close to reality.



Note that this idea of estimating values of derivatives can be a tricky process and does require a fair amount of (possible bad) approximations so while it can be used, you need to be careful with it.

We'll close out this section by noting that while we're not going to include an example here we could also use the graph of the derivative to give us a sketch of the function itself.