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THEORY OF FRAMES USE TO REDUCE THE INFORMATION LOSS IN SIGNALS TRANSMITTING

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***Summary**. The development of «smart city» conception must involve some new mathematical methods which allow us to reduce any information loss in data transfer in the smart city resource networks. Some possible use of advanced methods of the theory of frames for these problems has been taken into consideration in the article under discussion.*
***Key words**: smart city, information, signal, frame.*

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Problem statement. The article consists of three parts. In the first part some basic concepts of the theory of frames are taken into consideration. The theory of frames was described in papers more in detail [1, 2]. I must admit, that not all general statements of the theory of frames have been presented in the article under discussion, but only the results necessary for the problem solution, i.e. information loss reduction. These results will be used in the second and third parts to substantiate various capabilities of the theory of frames in the problems of noise reduction taking place during information transfer.

Paper aim. Development and efficient use of new mathematical methods (theory of frames) and approaches which will allow us to reduce the information loss in data transfer in the smart city resource networks.

Problem setting. We consider the frames specified in the space R^n. The elements of this space will be written as

$$f=(x_1,x_2,\dots x_n),\quad g=(y_1,y_2,\dots,y_n).$$

The measurement of this space is a natural number n, and the scalar product of random elements f and g is found as

$$(f,g)=\sum_{i=1}^{n}x_i y_i.$$

The vector norm f looks like $\| f\|^2=\sum_{i=1}^{n}x_i^2$.

Let us consider a set of vectors $\{f_k\}_{k=1}^m$ of the space R^n. The linear space of all possible combinations of vectors $\{f_k\}_{k=1}^m$, is called their linear hull, i.e.

$$span\{f_k\}_{k=1}^{m} = \{c_1 f_1 + c_2 f_2 + \ldots + c_m f_m\},$$

where $c_1, c_2, \ldots, c_m$ are random real numbers.

Definition 1. A random set of elements $\{f_k\}_{k=1}^{m}$ of the space R^n is called a frame, for which

$$span\{f_k\}_{k=1}^{m} = \mathrm{R}^n \quad (1)$$

According to the equality (1), the number of frame vectors m should be higher or equal to the space dimension n.

We must admit, that some vectors f_k on the frame definition can be zero vectors.

Definition 1 of the frame is rather simple to understand but not very convenient for practical use and theoretical investigations. In scientific literature, another equivalent definition of frame is usually used which is true for the frames in spaces of both finite and nonfinite dimensions.

Definition 2. A random series $\{f_k\}_{k=1}^{m}$ of the space elements R^n, for which nonzero numbers $0 < A \le B$ exist and those when $f \in \mathrm{R}^n$ are true inequalities, is called Frame

$$A\| f \|^2 \le \sum_{k=1}^{m} |(f, f_k)|^2 \le B\| f \|^2 . \quad (2)$$

Numbers A and B in (2) are called the boundaries of a frame. Maximum of all possible A and minimum of all possible B are called optimal boundaries of a frame.

Each vector $f \in \mathrm{R}^n$ can be taken into consideration as a signal. Let’s assume, that the frame $\{f_k\}_{k=1}^{m}$ was fixed. It «was analyzing» the signal $f \in \mathrm{R}^n$ by means of an analysis operator

$$Tf = ((f, f_1), (f, f_2), \ldots (f, f_m)),$$

which was acting from the space R^n to the space R^m.

Similarly, the operator of synthesis T^* of our frame $\{f_k\}_{k=1}^{m}$ acts from R^n to the space R^n and it is found by the following formula

$$T^* g = T^*\{c_k\}_{k=1}^{m} = \sum_{k=1}^{m} c_k f_k ,$$

where $g = \{c_k\}_{k=1}^{m} \in \mathbb{R}^m$ is a random vector of the space R^m. The definition of the synthesis operator in the form T^* underlines, that T^* is a conjugate operator relative to the analysis operator T. It means, that

$$(Tf, g)_{\mathbb{R}^m} = (f, T^* g)_{\mathbb{R}^n} \quad (3)$$

for all $f \in \mathrm{R}^n$ and for all $f \in \mathrm{R}^m$.

The equality (3) is a crucial one for our applications in the theory of noise restriction.

Combination of operators of analysis T and synthesis T^* is called a frame operator $\{f_k\}_{k=1}^m$

$$Sf = T^*Tf = \sum_{k=1}^{m}(f, f_k)f_k, \qquad f \in \mathbb{R}^n \tag{4}$$

From the equality (4) it follows, that

$$(Sf, f) = \sum_{k=1}^{m}|(f, f_k)|^2$$

The action of the frame operator S in the space R^n can be described by means of a positive square matrix S of the dimension n. We write symbols

$$\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$$

for all roots of a characteristic equation

$$\det[\mathbf{S} - \lambda I] = 0.$$

taking into account their multiplicity.

It is easy to see from (2), that the optimal boundaries A and B of the frame $\{f_k\}_{k=1}^m$ have coincided, respectively, with the smallest root λ_1 (the lowest value of the frame operator S) and the biggest root λ_n (the highest value of the frame operator S) of the characteristic level.

The properties of the frame operator allow us to prove one of the main results of the theory of frames.

Theorem. Let S is an operator of the frame $\{f_k\}_{k=1}^m$. Then the random element $f \in \mathrm{R}^n$ can be written as

$$f = \sum_{k=1}^{m}(f, S^{-1}f_k)f_k = \sum_{k=1}^{m}(f, f_k)S^{-1}f_k. \tag{5}$$

It is very easy to prove this important theorem. Actually, due to the acting on the inverse operator to the frame operator S^{-1} on the equality (4) we come to the second equality in (5). Then, we mark Sf as g . As a result, $f = S^{-1}g$ and the equality (4) can be written as follows

$$g = \sum_{k=1}^{m}(S^{-1}g, f_k)f_k = \sum_{k=1}^{m}(g, S^{-1}f_k)f_k,$$

where the vector g пробігає весь простір R^n, that has proved the first equality in (5).

From the *Theorem* follows, that a random frame $\{f_k\}_{k=1}^m$ allows us to show each vector of the space R^n, using the series of coefficients

$$c_k = (f, S^{-1} f_k), \qquad k = 1, \dots, m.$$

Though, this is a specific feature similar to that one, possessed by a random basis in R^n. So, the question has arisen: should we deal with frames?

The main difference between frames and bases is that in case of a random basis $\{\gamma\}_{k=1}^{n}$ any vector $f \in \mathrm{R}^n$ can be represented as a linear combination of basis vectors

$$f = \sum_{k=1}^{n} c_k \gamma_k , \tag{6}$$

where coefficients $\{c_k\}_{k=1}^{n}$ are undoubtfully defined by the vector f choice. This fact is already not true in case of frame which is not a basis (i.e. when $n < m$). This is an important distinctive feature, as unambiguity of coefficients $\{c_k\}$ in decomposition (6) is valuable from the theoretical point of view, it can lose its importance in practice. In the end, the very possibility to present a vector f is really important, but not the number of ways how to do it. Moreover, the requirement of the vector unique representation f is often found as a restrictive one. Bases may lose their properties due to even insignificant modifications of their elements. During information transfer a certain coefficient c_k can be lost, which in case of basis, contains some unique information of f (unique means that c_k cannot be expressed by other coefficients $c_j, j \neq k$ of decomposition (6)). Sometimes, it is impossible to build a basis obtaining certain features necessary for practical use. Besides, as it will be described further, the possibility of polyvalent choice of decomposition coefficients (6) allows us to reduce information loss. So, it is clear, that bases in Hilbert spaces lack some flexibility.

Frames can provide higher flexibility. They are a certain generalization of bases in Hilbert spaces. It is clear intuitively, that they can be considered as bases where some extra elements are added. This peculiarity, though it is seemed useless from the purely theoretical point of view, has been found to be very useful in practice.

Frame operator. We assume, that we would like to send a certain signal from the transmitter N to the receiver O. We represent this signal as a f, which is an element of the certain space R^n. If both on the level N, and on the level O we know something of the certain frame $\{f_k\}_{k=1}^{m}$ in space R^n (for example, we can perform calculations, dealing with this frame, using the appropriate library of programming), then thanks to (5) information about the signal f can be sent as a series of coefficients of the frame $\{(f, S^{-1} f_k)\}_{k=1}^{m}$.

Though, it should be mentioned, that we are interested in the implementation of such transmission in practice. Therefore, it is necessary to pay attention to some urgent problems occurring when we want to transfer a series of coefficients of the frame N to O. We may assume, that both on the level of the transmitter N and the receiver O, we deal with computer devices (for example, N and O can be represented by programs which operate on in-line computers in a parallel way and exchange information). The main difficulty is that the computers cannot store or transfer a sequence of real numbers – they are presented with a certain, restricted accuracy by means of rational numbers. That is why in most cases the coefficients of frame $\{(f, S^{-1} f_k)\}_{k=1}^m$ cannot be calculated accurately. Moreover, there is certainly a possibility of some extra disturbance caused by the very process of a signal f transmission and other technical aspects of the matter under consideration. Thanks to this, instead of the coefficient $(f, S^{-1} f_k)$, the receiver O obtains the following value

$$\left\langle f, S^{-1} f_k \right\rangle + w_k ,$$

where w_k is a certain error, or, in other words, noise (in this case we say, that the frame coefficients are disordered, or are polluted by noise $\{w_k\}_{k=1}^m$).

As it was found, frames allowed to reduce these negative consequences for the signal transmission quality and, in this way, allowed to reduce information loss in signals transmitting.

Use of dual frames for noise reduction in signals transmitting. In the space R^n we will consider the frame $\{f_k\}_{k=1}^m$. We assume, that the number m of the frame vectors is higher than the dimension n of the space R^n. It means, that the frame $\{f_k\}_{k=1}^m$ cannot be the basis of frame R^n and, what is the most important, there is unlimited number of frames $\{g_k\}_{k=1}^m$ of the space R^n таких, that each vector $f \in \mathrm{R}^n$ can be written as

$$f = \sum_{k=1}^{m} (f, g_k) f_k = \sum_{k=1}^{m} (f, f_k) g_k . \qquad (7)$$

A random frame $\{g_k\}_{k=1}^m$ of the space R^n is called a dual frame for the frame $\{f_k\}_{k=1}^m$ if the equality (7) is satisfied for it. According to the equality (5), it is clear, that the frame $\{S^{-1} f_k\}_{k=1}^m$ is an example of dual frame for $\{f_k\}_{k=1}^m$. This dual frame is called a canonical dual frame. A canonical dual frame is usually mostly used in applications, as it is rather easy to build it by means of a reverse operator to the frame S operator. The review of methods concerning general dual frames construction has been given in the paper [3].

Now we assume, that a transmitter N has a fixed frame $\{f_k\}_{k=1}^m$. Due to the frame the transmitter N is analyzing a random signal $f \in \mathrm{R}^n$, i.e. there is a series of numbers

$$(f_1, f_1), (f_1, f_2), \ldots, (f_1, f_m) .$$

This series of numbers is transferred to the receiver O. During this transfer some noise and some information loss appear, so the receiver O, in fact, obtains the value

$$(f, f_k) + w_k, \quad k = 1, \ldots, m ,$$

where w_k – is a certain error, or, in other words, some noise.

The receiver O knows about the fixed frame $\{f_k\}_{k=1}^{m}$ and, as a result, it can select a random dual frame $\{g_k\}_{k=1}^{m}$ to restore the signal by means of formula (7). Hence, the receiver O obtains somehow altered signal (information loss):

$$\tilde{f} = \sum_{k=1}^{m}((f, f_k) + w_k)g_k = \sum_{k=1}^{m}(f, f_k)g_k + \sum_{k=1}^{m} w_k g_k = f + \sum_{k=1}^{m} w_k g_k \qquad (8)$$

According to the equality (8), it follows, that the initial signal f differs from the altered signal $\tilde{f}$ in the following way

$$\sum_{k=1}^{m} w_k g_k. \qquad (9)$$

Is it possible to reduce the noise value (9) by means of a certain choice of the dual frame $\{g_k\}_{k=1}^{m}$?

To answer this question, we will take into consideration some additional results from the general theory of frames. We assume, that $\{g_k\}_{k=1}^{m}$ is a random frame in R^n. Then the expression (9) can be written as

$$T_g^* w = T_g^*\{w_k\} = \sum_{k=1}^{m} w_k g_k,$$

where T_g^* is the frame synthesis operator $\{g_k\}_{k=1}^{m}$, and the noise component w_k can be considered as vector $w = \{w_k\}$ in the space R^m. Hence, from the equality (3) it follows, that

$$T_g^*\{w_k\} = 0 \quad \Leftrightarrow \quad (T_g f, w)_{\mathbb{R}^m} = 0 \qquad \forall f \in \mathbb{R}^n,$$

where T_g is the frame analysis operator $\{g_k\}_{k=1}^{m}$.

Thus, noise (9) will disappear if the noise vector $w = \{w_k\}$ is an orthogonal one, relative to the set of images

$$\mathrm{R}(T_g) = \{T_g f = \{(f, g_k)\}_{k=1}^{m} : \forall f \in \mathbb{R}^n\} \qquad (10)$$

of the analysis operator T_g of the frame $\{g_k\}_{k=1}^{m}$.

Now, we can answer the question concerning the possibility of noise value reduction (9) by means of the dual frame $\{g_k\}_{k=1}^{m}$. In fact, we assume, that the receiver O knowns all types of disturbance which can cause some noise $w = \{w_k\}$. So, the receiver O can characterize a type of noise quite accurately. Interpreting this into the vector language in R^m – the receiver O can determine the noise subspace (error subspace) W in R^m, i.e. the subspace involving the noise vectors which can occur during the signal transfer. Choosing now such a dual frame

$\{g_k\}_{k=1}^{m}$ for which a set of images $\mathrm{R}(T_g)$ of the analysis operator T_g will be orthogonal to W, we can remove any noise completely.

We must admit, that in most applications it is rather doubtful that the noise can be «removed completely» as the receiver O won't be able to describe the noise subspace W absolutely accurately. Besides, there is a possibility, that a set of images $\mathrm{R}(T_g)$ cannot be found of the dual frame analysis operator which is orthogonal to W (but only almost orthogonal). Nevertheless, we can expect that the use of the above-mentioned technique will allow us to reduce the noise considerably (9) during the signal transfer.

Example. In the space B of the space R^2we will consider the frame Mercedes, which consists of vectors

$$f_1 = \sqrt{\frac{2}{3}}\begin{bmatrix}1\\0\end{bmatrix},\ f_2 = \sqrt{\frac{2}{3}}\begin{bmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{bmatrix},\ f_3 = \sqrt{\frac{2}{3}}\begin{bmatrix}-\frac{1}{2}\\ -\frac{\sqrt{3}}{2}\end{bmatrix}$$

Such name of the frame can be explained by its graphic representation which is similar to the company Mercedes-Bentz symbol (fig. 1)

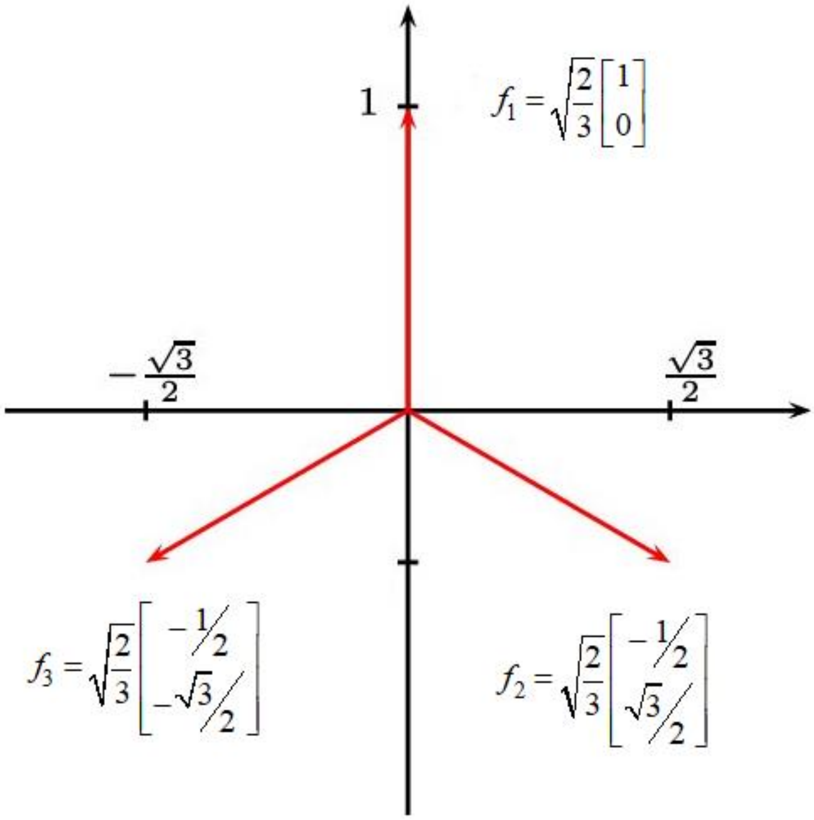

Figure 1. Mercedes frame

It is known [3], that all possible dual frames $\{g_k\}_{i=1}^{3}$ to the frame Mercedes look like

$$g_i = f_i + \frac{tg\,\alpha}{\sqrt{3}}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad , \tag{11}$$

and satisfy the requirements

$$0 \le \alpha < \frac{\pi}{2}$$

where u_1, u_2, α are random parameters and $u_1^2 + u_2^2 = 1$, and $0 \le \alpha < \dfrac{\pi}{2}$.

Based on the general theory (equality (7) each vector $f \in R^2$ can be written as

$$f = \sum_{i=1}^{3} (f, f_i)\, g_i$$

Besides, a set of images of the operator of analysis for the frame $\{g_k\}_{i=1}^{3}$ is a subspace in R^3

$$\left\{ \begin{bmatrix} (f, g_1) \\ (f, g_2) \\ (f, g_3) \end{bmatrix}, \quad \forall f \in \mathbb{R}^2 \right\} \tag{12}$$

When the information of the signal f is being transferred, the receiver O obtains some set of numbers

$$(f_1, f_1), (f_1, f_2), (f_1, f_3).$$

We have assumed, that the receiver O knows about the fact that the transfer of the second and the third values usually take place without any information loss (it means that the numbers (f_1, f_1) i (f_1, f_3) are exact ones). But the transfer of the first number is taking place incorrectly and the obtained number is not valid. Due to this information the receiver O describes the subspace of the noise in R^3, i.e.

$$W = \begin{bmatrix} w_1 \\ 0 \\ 0 \end{bmatrix}$$

We are searching for the correspondent frame which will oppose such type of pollution. The criterion of the good choice of frame is orthogonality of the subspace W to the subspace (12), i.e. the equality must be satisfied

$$(f, g_1)\, w_1 = 0 \text{ or } (f \;\; g_1) = \qquad \forall f \in \mathbb{R}^2 .$$

So, the best choice will be the frame $\{g_i\}_{i=1}^{3}$ where the first vector is equal to zero.

Using the formula (11) we have come to the conclusion that the parameters u_1, u_2, α should be chosen so that:

$$g_1 = \sqrt{\frac{2}{3}}\begin{bmatrix}1\\0\end{bmatrix} + \frac{tg\,\alpha}{\sqrt{3}}\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Hence,

$$\begin{cases}\sqrt{2} + tg\,\alpha\; u_1 = 0\\ u_2 = 0\end{cases}$$

It's easy to see that the solutions of the system will be $u_1 = -1$ i α=arctg $\sqrt{2}$.

Eventually, we get a dual frame $\{g_i\}$ making possible any noises to be removed in the first number of information transfer:

$$g_1 = 0, \quad g_2 = \sqrt{\frac{2}{3}}\begin{bmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{bmatrix} + \sqrt{\frac{2}{3}}\begin{bmatrix}-1\\0\end{bmatrix} = \sqrt{\frac{2}{3}}\begin{bmatrix}-\frac{3}{2}\\ \frac{\sqrt{3}}{2}\end{bmatrix}$$

$$g_3 = \sqrt{\frac{2}{3}}\begin{bmatrix}-\frac{1}{2}\\ -\frac{\sqrt{3}}{2}\end{bmatrix} + \sqrt{\frac{2}{3}}\begin{bmatrix}-1\\0\end{bmatrix} = \sqrt{\frac{2}{3}}\begin{bmatrix}-\frac{3}{2}\\ -\frac{\sqrt{3}}{2}\end{bmatrix}$$

Conclusions. Various capabilities of the theory of frames concerning the problems of noise reduction problems in information transfer have been substantiated in the paper under discussion. It should be stressed that the given result can be used in practice.

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ЗАСТОСУВАННЯ ТЕОРІЇ ФРЕЙМІВ ДЛЯ ЗМЕНШЕННЯ ВТРАТ ІНФОРМАЦІЇ ПРИ ПЕРЕДАВАННІ СИГНАЛІВ

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Резюме. *У дослідників та фахівців, що опрацьовують реальні інноваційні проекти для втілення в сучасних містах, чітко сформувались профілі концепту «Розумне місто». Це методологічний погляд на технологічно орієнтовану інформаційну та комунікаційну платформу, яка ефективно забезпечує реалізацію ключових обчислювальних алгоритмів та комплексів ІТ сервісів і системно інтегрує чисельні різнотипові пристрої, вбудовані в конкретні міські об'єкти. В обчислювальному середовищі проектів «розумних міст» фактично експлуатуються чисельні комплекси пристроїв, імплементованих у фізичні об'єкти, що підключені до мережі Інтернет. Вони, в свою чергу, підтримують набір різнотипових засобів зв'язку та протоколів обміну даними. Така системна інтеграція забезпечує ефективне надання широкого спектру послуг, які формуються завдяки об'єднанню як віртуальних, так і реальних фізичних пристроїв, інноваційних сервісів, що сформовані на базі сучасних інформаційних та комунікаційних технологій. Розвиток концепції «розумного міста» повинен містити нові математичні методи, які дозволяють зменшити втрати інформації при передаванні даних у ресурсних мережах розумного міста. В цій статті стисло описано можливості застосування сучасних метод теорії фреймів для зменшення втрат інформації при передаванні сигналів. Стаття побудована з трьох підрозділів. У першому розглядаються принципові поняття теорії фреймів. Детально теорія фреймів описана в роботах Christensen O. та C. Heil. У роботі не представлено всі загальні положення теорії фреймів, а лише результати, необхідні для розв'язування задачі – зменшення втрати інформації. Наведено відповідні означення та теореми. Отримані результати використано в наступних підрозділах статті для обґрунтування різноманітних можливостей теорії фреймів у задачах зменшення шуму, який з'являється при передаванні інформації. Фрейми можуть забезпечити більшу гнучкість. Вони становлять певне узагальнення базисів у гільбертових просторах. Інтуїтивно їх можна розглядати як базиси, до яких додано зайві елементи. Ця властивість, хоч може здатися зайвою з чисто теоретичних міркувань, виявляється корисною на практиці.*

Ключові слова: *розумне місто, інформація, сигнал, фрейм.*

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