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Методичні вказівки

для самостійної роботи з дисципліни

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Department of Mathematical Methods in Engineering

Methodical instructions

for self study of students of all forms of studies
with the

«Elements of Linear Algebra»

of Higher Mathematics course

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Contents

Chapter 1. Matrices and matrix operations

§1.1. Matrices: essential definitions

<u>*Definition*</u>. *A* ***matrix (matrices*** in plural**)** is a rectangular array of numbers, symbols or expressions, arranged in rows and columns and written within parentheses. The items in a matrix are called the ***entries*** or ***elements***.

If every entry of a matrix is a number, the matrix is called a number matrix.

In this course of linear algebra we will consider number matrices.

An example of the number matrix with 3 rows and 4 columns is

$$\begin{array}{r c} & \begin{array}{cccc} \text{Column 1} & \text{Column 2} & \text{Column 3} & \text{Column 4} \end{array} \\ \begin{array}{r} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \end{array} & \begin{pmatrix} 1 & 2 & 4 & 7 \\ -1 & 3 & 0 & 5 \\ -2 & 5 & 6 & -4 \end{pmatrix} \end{array} \tag{1.1}$$

Numbers 1, 2, 4, 7, −1, 3, etc. are entries of this matrix.

The rows of a matrix are numbered from the top to the bottom and the columns are numbered from left to right.

Matrices are denoted using upper-case letters, while the corresponding lower-case letters with subscripts represent the entries.

If A is a matrix with m rows and n columns we say A is an $m \times n$ (m by n) matrix. Also the matrix A is said to be $m \times n$ dimension or $m \times n$ size or $m \times n$ order.

The matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \tag{1.2}$$

has m rows and n columns, so it is an $m \times n$ matrix.

Note, that the first number of subscripts on an element corresponds to the row, and the second number corresponds to the column in which the entry is located.

Thus $a_{i\,j}$ (a sub i, j) represents the entry in the i-th row and j-th column. In the matrix (1.1) above, the entry $a_{14} = 7$, $a_{23} = 0$.

A common more brief notation is $A_{m\times n} = \left(a_{i\,j}\right)$ or $A = \left(a_{i\,j}\right)$, $i = 1,..,m,\ j = 1,...,n$.

Some matrices take special names because of their size.

<u>*Definition*</u>. *A* matrix with only one column is called a ***column matrix***, and one with only one row is called a ***row matrix***.

<u>*Definition*</u>. *A* ***square matrix*** is a matrix with the same number of rows and columns.

An n-by-n matrix is known as a square matrix of order n.

The elements a_{11}, a_{22}, …, a_{nn} form the ***principal (main) diagonal*** of a square matrix.

These definitions are illustrated by the following:

$\begin{pmatrix} 3 \\ 2 \\ -1 \\ 4 \end{pmatrix}$ 4 × 1 column matrix; $\begin{pmatrix} -1 & 5 & 7 \end{pmatrix}$ 1 × 3 row matrix;

$\begin{pmatrix} 1 & -2 & 0 \\ 2 & 4 & -5 \\ 3 & 7 & 9 \end{pmatrix}$ 3 × 3 square matrix. The elements 1, 4, 9 form the main diagonal.

<u>*Definition*</u>. A square matrix, in which all elements outside the main diagonal are zero, is called a ***diagonal matrix***.

<u>*Definition*</u>. A square matrix, in which all entries above (below) the main diagonal are zero, is called a ***lower-triangular matrix*** (***upper-triangular matrix***), respectively.

For example, for square matrix of order 3, they look like

$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$ − diagonal matrix;

$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ − lower triangular matrix;

$\begin{pmatrix} a_{11} & a_{1} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$ − upper triangular matrix.

<u>*Definition*</u>. Any matrix in which every entry is zero is called a ***zero matrix*** (denoted O). Examples are $O_{2\times2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; $O_{2\times3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

<u>*Definition*</u>. A diagonal matrix in which every entry of the main diagonal is one is called a ***unit matrix*** or ***identity matrix*** (denoted I).

Examples are $I_{2\times2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $I_{3\times3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Matrices can’t be compared because they differ both dimensions and entries.

But we can define equality of two matrices.

<u>*Definition*</u>. Two matrices $A = \left(a_{ij}\right)$ and $B = \left(b_{ij}\right)$ are equal if and only if:

1) A and B are the same dimension;
2) $a_{ij} = b_{ij}$ for all i and j.

Example 1.1. Determine the dimension of the matrix $A = \begin{pmatrix} 2 & 3 & 1 \\ -1 & 0 & 4 \end{pmatrix}$ and write its entries.

Solution. Matrix $A = \begin{pmatrix} 2 & 3 & 1 \\ -1 & 0 & 4 \end{pmatrix}$ has the size 2×3, because it consists of two rows and three columns:

$$\begin{array}{r c} & \begin{array}{ccc} \text{Column 1} & \text{Column 2} & \text{Column 3} \end{array} \\ \begin{array}{r} \text{Row 1} \\ \text{Row 2} \end{array} & \begin{pmatrix} 2 & 3 & 1 \\ -1 & 0 & 4 \end{pmatrix} \end{array}$$

Entries: $a_{11} = 2$, $a_{12} = 3$, $a_{13} = 1$; $a_{21} = -1$, $a_{22} = 0$, $a_{23} = 4$.

§1.2. Transpose of a matrix. Linear matrix operations

Transpose of a matrix

The matrix unary operation is the transpose of a matrix.

Definition. If A is $m \times n$ matrix then the ***transpose of A***, denoted by A^{T}, is $n \times m$ matrix that is obtained by interchanging the rows and columns of A:

$$A_{m\times n} = \left(a_{ij}\right) \Rightarrow A^T_{n\times m} = \left(a_{ji}\right) \text{ for } i = 1,\ldots,m, \quad j = 1,\ldots,n. \tag{1.3}$$

In other words, the first row A^{T} is the first column of A, the second row of A^{T} is the second column of A, etc.

Example 1.2. Determine the transpose for the following matrices.

$$A = \begin{pmatrix} 4 & -5 & 3 \\ 2 & 7 & 0 \end{pmatrix}; \; B = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}; \; C = \begin{pmatrix} 1 & 2 \\ 5 & -7 \end{pmatrix}.$$

Solution.

$$A^T = \begin{pmatrix} 4 & 2 \\ -5 & 7 \\ 3 & 0 \end{pmatrix}; \; B^T = \begin{pmatrix} 3 & -1 & 4 \end{pmatrix}; \; C^T = \begin{pmatrix} 1 & 5 \\ 2 & -7 \end{pmatrix}.$$

Matrix Addition

Definition. If matrix $A = \left(a_{ij}\right)$ and $B = \left(b_{ij}\right)$ have the same order $m \times n$ then their sum $A + B$ (difference $A - B$) is a new $m \times n$ matrix $C = \left(c_{ij}\right)$ in which each entry is a sum (difference) of the corresponding entries of matrices A and B.

Thus

$$C = A + B \quad \Leftrightarrow \quad c_{ij} = a_{ij} + b_{ij} \quad \text{for} \quad i = 1,\ldots,m, \quad j = 1,\ldots,n; \tag{1.4}$$

$$C = A - B \quad \Leftrightarrow \quad c_{ij} = a_{ij} - b_{ij} \quad \text{for} \quad i = 1,\ldots,m, \quad j = 1,\ldots,n. \tag{1.5}$$

<u>*Note.*</u> Matrices of different sizes cannot be added or subtracted.

<u>*Example 1.3.*</u> For matrices $A = \begin{pmatrix} 4 & 5 & -3 \\ 2 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 6 & 10 \\ 5 & -4 & 3 \end{pmatrix}$ find:

1) the sum of A and B ;
2) the difference $A - B$.

<u>*Solution.*</u>

1) $C = A + B = \begin{pmatrix} 4+7 & 5+6 & -3+10 \\ 2+5 & 1+(-4) & 0+3 \end{pmatrix} = \begin{pmatrix} 11 & 11 & 7 \\ 7 & -3 & 3 \end{pmatrix}$;

2) $C = A - B = \begin{pmatrix} 4-7 & 5-6 & -3-10 \\ 2-5 & 1-(-4) & 0-3 \end{pmatrix} = \begin{pmatrix} -3 & -1 & -13 \\ -3 & 5 & -3 \end{pmatrix}$.

Scalar Multiplication

<u>*Definition*</u>. The product of a number k (called a scalar) by a matrix $A_{m\times n} = \left(a_{ij}\right)$ is a new matrix $C_{m\times n} = \left(c_{ij}\right)$ in which each entry of A is multiplied by k .

Thus

$$C = k \cdot A \quad \Leftrightarrow \quad c_{ij} = k \cdot a_{ij} \quad \text{for} \quad i = 1,\ldots,m, \quad j = 1,\ldots,n. \tag{1.6}$$

The matrix $\left(-1\right) \cdot A = -A$ is called the negative of the matrix A.

<u>*Example 1.4.*</u> If $N = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 5 & 4 \\ -7 & 2 & 6 \end{pmatrix}$ find $5N$ and $-3N$.

<u>*Solution.*</u>

$$5N = 5 \cdot \begin{pmatrix} 1 & -2 & 3 \\ 0 & 5 & 4 \\ -7 & 2 & 6 \end{pmatrix} = \begin{pmatrix} 5\cdot 1 & 5\cdot\left(-2\right) & 5\cdot 3 \\ 5\cdot 0 & 5\cdot 5 & 5\cdot 4 \\ 5\cdot\left(-7\right) & 5\cdot 2 & 5\cdot 6 \end{pmatrix} = \begin{pmatrix} 5 & -10 & 15 \\ 0 & 25 & 20 \\ -35 & 10 & 30 \end{pmatrix};$$

$$-3N = -3 \cdot \begin{pmatrix} 1 & -2 & 3 \\ 0 & 5 & 4 \\ -7 & 2 & 6 \end{pmatrix} = \begin{pmatrix} -3\cdot 1 & -3\cdot\left(-2\right) & -3\cdot 3 \\ -3\cdot 0 & -3\cdot 5 & -3\cdot 4 \\ -3\cdot\left(-7\right) & -3\cdot 2 & -3\cdot 6 \end{pmatrix} = \begin{pmatrix} -3 & 6 & -9 \\ 0 & -15 & -12 \\ 21 & -6 & -18 \end{pmatrix}.$$

Example 1.5. Let $A=\begin{pmatrix}1 & -3\\ 2 & 4\end{pmatrix}$, $B=\begin{pmatrix}8 & 6\\ -4 & 0\end{pmatrix}$. Find matrix $C=2\left(A-5I\right)+3B^T$.

Solution.

$$C=2\cdot\left(\begin{pmatrix}1 & -3\\ 2 & 4\end{pmatrix}-5\cdot\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)+3\cdot\begin{pmatrix}8 & 6\\ -4 & 0\end{pmatrix}^T=$$

$$=2\cdot\left(\begin{pmatrix}1 & -3\\ 2 & 4\end{pmatrix}-\begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix}\right)+3\cdot\begin{pmatrix}8 & -4\\ 6 & 0\end{pmatrix}=2\cdot\begin{pmatrix}-4 & -3\\ 2 & -1\end{pmatrix}+3\cdot\begin{pmatrix}8 & -4\\ 6 & 0\end{pmatrix}=$$

$$=\begin{pmatrix}-8 & -6\\ 4 & -2\end{pmatrix}+\begin{pmatrix}24 & -12\\ 18 & 0\end{pmatrix}=\begin{pmatrix}16 & -18\\ 22 & -2\end{pmatrix}.$$

§1.3. Matrix multiplication

Definition. If $A=\left(a_{ij}\right)$ is an $m\times n$ matrix and $B=\left(b_{ij}\right)$ is an $n\times p$ matrix then their product AB is a new $m\times p$ matrix $C=\left(c_{ij}\right)$ in which each entry is given by the formula

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{in}b_{nj}=\sum_{l=1}^{n}a_{il}b_{lj}\ ,\ \text{for}\ \ i=1,\ldots,m,\quad j=1,\ldots,n. \qquad (1.7)$$

This formula is illustrated in Figure 1.1.

$$AB=\begin{pmatrix} & \vdots & \\ a_{i1} & a_{i2} \ \cdots & a_{in} \\ & \vdots & \end{pmatrix}\begin{pmatrix} & b_{1j} & \\ \cdots & b_{2j} & \cdots \\ & \vdots & \\ & b_{nj} & \end{pmatrix}=\begin{pmatrix} & \vdots & \\ \cdots & \boxed{c_{ij}} & \cdots \\ & \vdots & \end{pmatrix}$$

Figure 1.1.

The product A and B is said to be made by product the rows of the first matrix and the columns of the second matrix.

Note that the product of two matrices can be obtained if and only if the number of columns of the first matrix is equal to the number of rows of the second one (consistency condition).

Use this fact to check quickly whether a given multiplication is defined. Write the product in terms of the matrix dimensions. In the case of the our definition, A is $m\times n$ and B is $n\times p$, so AB is $(m\times n)(n\times p)$. The middle values match:

$$(m\times \overbrace{n)(n}^{\textit{product is defined}}\times p)$$

so the multiplication is defined. By the way, you will recall that *AB*, the product matrix, was $m \times p$. You can also see this on the dimensions (external values):

$$\underbrace{(m\times n)(n\times p)}_{\textit{product will be } m\times p}$$

<u>*Example 1.6.*</u> Find AB if $A=\begin{pmatrix}3 & 5\\ 7 & 9\\ 4 & 2\end{pmatrix}$ and $B=\begin{pmatrix}a & b & c & d\\ e & f & g & h\end{pmatrix}$.

<u>*Solution.*</u> A is an 3×2 matrix and B is an 2×4 matrix, so the number of columns of A equals the number of rows of B. Thus we can find the product *AB*, which is a 3×4 matrix:

$$\underbrace{(3\times \overbrace{2)(2}^{\textit{product is defined}}\times 4)}_{\textit{product will be } 3\times 4}$$

$$AB=\begin{pmatrix}3 & 5\\ 7 & 9\\ 4 & 2\end{pmatrix}\begin{pmatrix}a & b & c & d\\ e & f & g & h\end{pmatrix}=\begin{pmatrix}3a+5e & 3b+5f & 3c+5g & 3d+5h\\ 7a+9e & 7b+9f & 7c+9g & 7d+9h\\ 4a+2e & 4b+2f & 4c+2g & 4d+2h\end{pmatrix}.$$

Note that the product *BA* is not defined, because the consistency condition is not satisfied:

$$(2\times \overbrace{4)(3}^{\textit{product is not defined}}\times 2)$$

But the product of two square $n \times n$ matrices is always defined.

<u>*Example 1.7.*</u> Find *AB* and *BA* if $A=\begin{pmatrix}2 & 3\\ 0 & 1\end{pmatrix}$ and $B=\begin{pmatrix}1 & 3\\ 2 & 1\end{pmatrix}$.

<u>*Solution.*</u> Both A and B are 2×2 matrices, so both *AB* and *BA* are defined.

$$AB=\begin{pmatrix}2 & 3\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 3\\ 2 & 1\end{pmatrix}=\begin{pmatrix}2\cdot 1+3\cdot 2 & 2\cdot 3+3\cdot 1\\ 0\cdot 1+1\cdot 2 & 0\cdot 3+1\cdot 1\end{pmatrix}=\begin{pmatrix}8 & 9\\ 2 & 1\end{pmatrix}$$

$$BA=\begin{pmatrix}1 & 3\\ 2 & 1\end{pmatrix}\begin{pmatrix}2 & 3\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1\cdot 2+3\cdot 0 & 1\cdot 3+3\cdot 1\\ 2\cdot 2+1\cdot 0 & 2\cdot 3+1\cdot 1\end{pmatrix}=\begin{pmatrix}2 & 6\\ 4 & 7\end{pmatrix}$$

Note that for these matrices the two products *AB* and *BA* are the same size matrices, but they are not equal. That is $AB \neq BA$. Thus Matrix multiplication is not commutative.

Definition. The power A^n of a square matrix A for n a nonnegative integer is defined as the matrix product of n copies of A,

$$A^n = \underbrace{A\cdot A\cdot \ldots \cdot A}_{n \text{ times}}. \tag{1.8}$$

Example 1.8. Given matrices $A=\begin{pmatrix}2&3\\0&1\\4&-2\end{pmatrix}$, $B=\begin{pmatrix}5&1\\6&-3\\7&0\end{pmatrix}$, $C=\begin{pmatrix}8&9\\-4&0\end{pmatrix}$, $D=\begin{pmatrix}7\\2\\0\\4\end{pmatrix}$, $G=\begin{pmatrix}-2&1\end{pmatrix}$, I – identity matrix. Perform the indicated operations or explain why it is not possible:

1) $A\cdot C$; 2) $B\cdot D$; 3) $D\cdot G$; 4) $(C-3I)\cdot G^T$; 5) $A^T\cdot B+3C^2$; 6) D^2.

Solution.

1) Firstly check whether a given multiplication is defined. Write the product in terms of the matrix dimensions: $\underbrace{(3\times \overbrace{2)(2}^{\textit{product is defined}}\times 2)}_{\textit{product will be } 3\times 2}$, therefore calculate

$$A\cdot C=\begin{pmatrix}2&3\\0&1\\4&-2\end{pmatrix}\cdot\begin{pmatrix}8&9\\-4&0\end{pmatrix}=\begin{pmatrix}2\cdot 8+3\cdot(-4)&2\cdot 9+3\cdot 0\\0\cdot 8+1\cdot(-4)&0\cdot 9+1\cdot 0\\4\cdot 8-2\cdot(-4)&4\cdot 9-2\cdot 0\end{pmatrix}=\begin{pmatrix}4&18\\-4&0\\40&36\end{pmatrix};$$

2) $B\cdot D=\begin{pmatrix}5&1\\6&-3\\7&0\end{pmatrix}\cdot\begin{pmatrix}7\\2\\0\\4\end{pmatrix}$ – the product is not defined, because the columns of B aren't the same length as the rows of D $(2\neq 4)$: $(3\times \overbrace{2)(4}^{\textit{product is not defined}}\times 1)$;

3) $\underbrace{(4\times \overbrace{1)(1}^{\textit{product is defined}}\times 2)}_{\textit{product will be } 4\times 2}$,therefore $D\cdot G=\begin{pmatrix}7\\2\\0\\4\end{pmatrix}\cdot\begin{pmatrix}-2&1\end{pmatrix}=\begin{pmatrix}7\cdot(-2)&7\cdot 1\\2\cdot(-2)&2\cdot 1\\0\cdot(-2)&0\cdot 1\\4\cdot(-2)&4\cdot 1\end{pmatrix}=\begin{pmatrix}-14&7\\-4&2\\0&0\\-8&4\end{pmatrix}$;

4) firstly perform action in brackets:

$$\left(C-3I\right)=\begin{pmatrix}8&9\\-4&0\end{pmatrix}-3\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}8&9\\-4&0\end{pmatrix}-\begin{pmatrix}3&0\\0&3\end{pmatrix}=\begin{pmatrix}5&9\\-4&-3\end{pmatrix};\ \text{then}$$

$$\left(C-3I\right)\cdot G^T=\begin{pmatrix}5&9\\-4&-3\end{pmatrix}\cdot\begin{pmatrix}-2\\1\end{pmatrix}=\begin{pmatrix}5\cdot(-2)+9\cdot 1\\(-4)\cdot(-2)-3\cdot 1\end{pmatrix}=\begin{pmatrix}-1\\5\end{pmatrix};$$

5) perform step by step:

$$A^T\cdot B=\begin{pmatrix}2&0&4\\3&1&-2\end{pmatrix}\cdot\begin{pmatrix}5&1\\6&-3\\7&0\end{pmatrix}=\begin{pmatrix}10+0+28&2+0+0\\15+6-14&3-3+0\end{pmatrix}=\begin{pmatrix}38&2\\7&0\end{pmatrix};$$

$$3C^2=3\cdot\begin{pmatrix}8&9\\-4&0\end{pmatrix}\cdot\begin{pmatrix}8&9\\-4&0\end{pmatrix}=3\cdot\begin{pmatrix}64-36&72+0\\-32+0&-36+0\end{pmatrix}=\begin{pmatrix}84&216\\-96&-108\end{pmatrix};$$

$$A^T\cdot B+3C^2=\begin{pmatrix}38&2\\7&0\end{pmatrix}+\begin{pmatrix}84&216\\-96&-108\end{pmatrix}=\begin{pmatrix}122&218\\-89&-108\end{pmatrix}.$$

6) D^2 – cannot be performed, because the power A^n of a matrix A is defined only for a square matrix.

§1.4. Properties of Matrix Operations

Operation	Description	Property	Name of property
Addition	Let A,B,C be $m\times n$ matrices, O be zero matrix of the same size	$A+B=B+A$ $(A+B)+C=A+(B+C)$ $A+O=A$ $A+(-A)=O$	commutative property associative property additive identity property additive inverse property
Scalar Multiplication	Let A,B be $m\times n$ matrices, c and d be scalars	$(cd)A=c\,(dA)$ $(c+d)A=cA+dA$ $c\,(A+B)=cA+cB$	 distributive property distributive property
Matrix multiplication	Let A,B and C be matrices with sizes such that the regarded operations are defined, I be the identity matrix, c be a scalar	$(AB)C=A(BC)$ $A(B+C)=AB+AC$ $(B+C)A=BA+CA$ $c\,(AB)=(cA)B=A(cB)$ $A_{m\times n}\cdot I_{n\times n}=A_{m\times n}$ and $I_{m\times m}\cdot A_{m\times n}=A_{m\times n}$	associative property distributive property distributive property multiplicative identity

Transpose	Let A,B be matrices with sizes such that the regarded operations are defined, c be a scalar	$\left(A^T\right)^T = A$ $(A+B)^T = A^T + B^T$ $(c\cdot A)^T = c\cdot A^T$ $(A\cdot B)^T = B^T \cdot A^T$	

Check yourself

1. The dimension of matrix A is 5×3. How many rows and columns this matrix has?
2. Suppose A and B are matrices of the same size. Does it make sense to write $A > B$?
3. Give the definition of linear matrix operations.
4. Let A be 6×4, B be 4×4, C be 4×3, D and E be 3×1. Determine which of the following are defined, and for those that are, give the size of the resulting matrix.

a) A^{10}

b) $BC + CB$

c) ABC

d) B^{10}

e) $ACBD$

f) $ABCD$

g) $C(2E - D)$

h) $CD + E$

5. Is it true that if $A\cdot B = 0$, for matrices A and B , then $A = 0$ or $B = 0$? Why or why not?
6. What matrices can be added?
7. How is called the diagonal matrix in which all nonzero entries are one?
8. Formulate the consistency condition for matrix multiplication.
9. Give the definition of the upper triangular matrix.
10. How is called the matrix in which the numbers of rows and columns are the same?
11. Explain why the commutative property for matrix product is not satisfy. Give an example.
12. Does it make sense to calculate A^3 if matrix A is:
    a) an $m\times n$ dimension;
    b) a square matrix of n-th order.
13. Give the definition of the identity matrix.
14. Formulate the properties of scalar multiplication.
15. Formulate the properties of matrix multiplication.
16. Formulate the properties of transpose.
17. Formulate the properties of matrix addition.
18. Give an example of matrices A and B such that neither A no B equals zero and yet $AB = 0$.
19. Suppose A and B are square matrices of the same size. Which of the following are correct?

a) $\left(A-B\right)^2 = A^2 - 2AB + B^2$

b) $A^2B^2 = A\left(AB\right)B$

c) $\left(A+B\right)^2 = A^2 + AB + BA + B^2$

d) $A^2B^2 = \left(AB\right)^2$

e) $\left(A+B\right)\left(A-B\right) = A^2 - B^2$

20.Give an example of a matrix A such that $A^2 = I$ and yet $A \neq I$ and $A \neq -I$.

Problem set

1. Given matrices: $A = \begin{pmatrix} 2 & -3 \\ 4 & -1 \\ 3 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 6 \\ -1 & 0 \\ 2 & -1 \end{pmatrix}$. Perform the indicated operations:

a) $A+B$; b) $A-B$, c) $-4A$.

2. Given matrices: $A = \begin{pmatrix} 2 & -4 \\ 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 5 \\ 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$. Perform the operations on matrices:

a) $3A + B^T$; b) AB; c) BA; d) $3AC$; e) C^TB; f) CA.

3. Given matrices: $A = \begin{pmatrix} 0 & -3 & 1 \\ 2 & 1 & 5 \\ -4 & 0 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$. Find the product *AB*.

4. Find the products MG and GM, if $M = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & 4 \\ 1 & 2 & 3 \end{pmatrix}$ and $G = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \\ 3 & 2 & 1 \end{pmatrix}$.

5. Perform the operations on matrices: a) $\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}^3$; b) $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}^2$.

6. Given matrices: $A = \begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$. Verify the property $\left(A \cdot B\right)^T = B^T \cdot A^T$ directly for these matrices.

Answers

1. a) $\begin{pmatrix} 7 & 3 \\ 3 & -1 \\ 5 & 4 \end{pmatrix}$; b) $\begin{pmatrix} -3 & -9 \\ 5 & -1 \\ 1 & 6 \end{pmatrix}$; c) $\begin{pmatrix} -8 & 12 \\ -16 & 4 \\ -12 & -20 \end{pmatrix}$.

2. a) $\begin{pmatrix} 4 & -11 \\ 14 & 3 \end{pmatrix}$; b) $\begin{pmatrix} -8 & 10 \\ -5 & 15 \end{pmatrix}$; c) $\begin{pmatrix} 11 & 13 \\ 2 & -4 \end{pmatrix}$; d) $\begin{pmatrix} 102 \\ 6 \end{pmatrix}$; e) $\begin{pmatrix} -13 & 15 \end{pmatrix}$; f) impossible.

3. $\begin{pmatrix} 8 \\ 14 \\ 16 \end{pmatrix}$.

4. $MG = \begin{pmatrix} 8 & 0 & 7 \\ 16 & 10 & 4 \\ 13 & 5 & 7 \end{pmatrix}$, $GM = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 7 & 3 \\ 8 & 11 & 14 \end{pmatrix}$.

5. a) $\begin{pmatrix} 13 & -14 \\ 21 & -22 \end{pmatrix}$; b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Chapter 2. Determinant of a Matrix. Inverse of a Matrix. Rank of a Matrix

§2.1. Determinant of a Matrix

Definition. If A is a square matrix, then the ***determinant of*** A is a number evaluated by the certain formula from the entries of this matrix. The determinant of A is denoted by **|*A*|** or **det *A*** or simply by writing the array of entries in A withing vertical lines:

$$\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Evaluating Determinant of the first order. The determinant of a 1×1 matrix $A = (a_{11})$ is simply the element a_{11} itself:

$$det\,A = |a_{11}| = a_{11}\ . \tag{2.1}$$

Caution. In the present context, the braces around a_{11}, in the middle member of (2.1), denote determinant, no absolute value. For instance, if $A = (-7)$, then $det\,A = |-7| = -7$.

Evaluating Determinant of the second order. The determinant of a 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is a real number evaluated by the formula

$$det\,A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{2.2}$$

Otherwise speaking a second order determinant is the product of the principal diagonal entries (from upper left to lower right) minus the product of the secondary diagonal entries (from upper right to lower left).

Evaluating Determinant of the third order. The determinant of a 3×3 matrix

$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is a real number evaluated by the formula

$$det\,A=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=$$

$$=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{21}a_{32}a_{13}-a_{31}a_{22}a_{13}-a_{21}a_{12}a_{33}-a_{32}a_{23}a_{11} \qquad (2.3)$$

To memorize the last formula used **Triangle's rule or Sarrus's rule.**
The triangle's rule will be formed with this figure 2.1:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}-$$

$$-\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}-\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}-\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Figure 2.1

The product of principal diagonal elements and product of elements in the vertex of the first and the second triangles get the "+" sign and the product of secondary diagonal elements and product of elements in the vertex of the third and the fourth triangles get the "–" sign. In base of triangle's rule, we have:

$$det\,A=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}-\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=$$

$$=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{21}a_{32}a_{13}-a_{31}a_{22}a_{13}-a_{21}a_{12}a_{33}-a_{32}a_{23}a_{11}.$$

The Sarrus's rule will be formed with those figure 2.2 and figure 2.3:

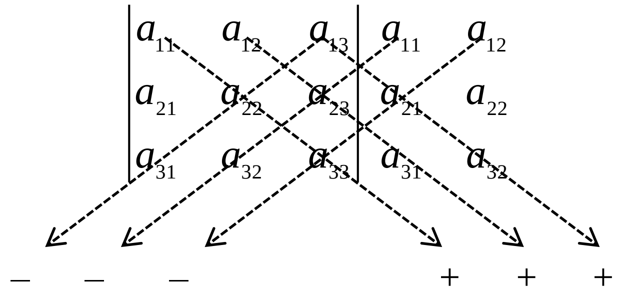


Figure 2.2

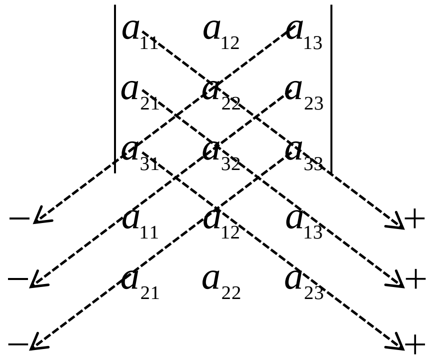


Figure 2.3

From the description of two first columns of the determinants (first and second columns) will be formed Figure 2.2 respectively two rows (first and second rows) will be formed Figure 2.3. The terms, which will be formed by the products of elements along the diagonals from upper left to lower right in both of Figure 2.2 and Figure 2.3 become the “+“ sign but products along the diagonals from upper right to lower left become the “–“ sign.

Example 2.1. Evaluate the determinants:

a) $\begin{vmatrix} 2 & 4 \\ 3 & -5 \end{vmatrix}$; b) $\begin{vmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{vmatrix}$; c) $\begin{vmatrix} 3 & 1 & -7 \\ 4 & 0 & 2 \\ 5 & 6 & 10 \end{vmatrix}$.

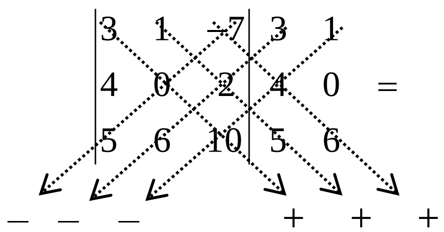
Solution.

a) Using the formula (2.2) the determinant can be found

$$\begin{vmatrix} 2 & 4 \\ 3 & -5 \end{vmatrix} = 2\cdot(-5) - 3\cdot 4 = -10 - 12 = -22;$$

b) $$\begin{vmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{vmatrix} = \sin\alpha\cdot\sin\alpha - \cos\alpha\cdot(-\cos\alpha) = \sin^2\alpha + \cos^2\alpha = 1;$$

c) In base of the Sarrus’s rule we have:



$$3\cdot 0\cdot 10 + 1\cdot 2\cdot 5 + (-7)\cdot 4\cdot 6 - (-7)\cdot 0\cdot 5 - 3\cdot 2\cdot 6 - 1\cdot 4\cdot 10 =$$

$$= 0 + 10 - 168 + 0 - 36 - 40 = -234.$$

§2.2. Properties of Determinant

The determinant has the following properties.

1. Let A be a square matrix. If A and A^T are transposes of each other, then $\det A = \det A^T$. Otherwise speaking if rows of a determinant interchange into columns then the new determinant is equal to the old one. That is why all the properties of determinant which is formulated for rows hold try also for columns.

2. If two rows (or two columns) of a determinant are interchanged, the new determinant is the negative of the old.

   For example, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{vmatrix}$. Here the first and the second columns are interchanged.

3. A factor common to all elements of a row (or column) can be taken out as a factor of the determinant.

   For example, $\begin{vmatrix} a_{11} & a_{12} & ka_{13} \\ a_{21} & a_{22} & ka_{23} \\ a_{31} & a_{32} & ka_{33} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

4. If every element in a row (or column) of a determinant is zero, the value of the determinant is zero.

   For example, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$ or $\begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix} = 0$.

5. If the corresponding elements are equal in two rows (or columns), the value of the determinant is zero.

   For example, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a & b & c \\ a & b & c \end{vmatrix} = 0$.

6. If one row (or column) of a determinant is a multiple of another row (or column) of the determinant then the determinant is zero.

   For example, $\begin{vmatrix} ka & kb & kc \\ a_{21} & a_{22} & a_{23} \\ a & b & c \end{vmatrix} = 0$. Here $R_1 = k \times R_3$.

7. If every entry of one row (or column) is represented in the form of sum of two addends then the determinant is equal to sum of two determinants: in the first determinant entries of the row (or column) are the first addends, in the second determinant entries of the one row (or column) are the second addends.

For example, $\begin{vmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$

8. If a multiple of any row (or column) of a determinant is added to any other row (or column), the value of the determinant is not changed.

For example, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+ka_{13} & a_{12} & a_{13} \\ a_{21}+ka_{23} & a_{22} & a_{23} \\ a_{31}+ka_{33} & a_{32} & a_{33} \end{vmatrix}.$

9. The determinant of a diagonal matrix is the product of its diagonal entries.

For example, $\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11} \cdot a_{22} \cdot a_{33}.$

10. The determinant of a lower-triangular matrix or upper-triangular matrix is the product of its diagonal entries.

For example, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11} \cdot a_{22} \cdot a_{33}.$

11. The determinant of a product of two matrices is equal to a product of their determinants.

$$\det(A \cdot B) = \det A \cdot \det B.$$

These properties can be used to facilitate the computation of determinants by simplifying the matrix to the point where the determinant can be determined immediately. Specifically the property 8 can be used to transform any matrix into a triangular matrix, whose determinant is given by the property 10.

Example 2.2. Evaluate the determinant using properties of determinant

$$\begin{vmatrix} 2 & 0 & -2 \\ -2 & 1 & 3 \\ 3 & -2 & 1 \end{vmatrix}.$$

Solution.

First, we add the first column to the third column (property 8):

$$\begin{vmatrix} 2 & 0 & -2 \\ -2 & 1 & 3 \\ 3 & -2 & 1 \end{vmatrix} \begin{array}{l} \textit{Column 1 is unchanged} \\ \textit{Column 2 is unchanged} \\ Col1 + Col3 = New\ Col3 \end{array} = \begin{vmatrix} 2 & 0 & 0 \\ -2 & 1 & 1 \\ 3 & -2 & 4 \end{vmatrix} \quad \begin{array}{l} \textit{Henceforth we will write} \\ \textit{more briefly} \\ C_1 + C_3 \to C_3 \end{array}$$

Next, we multiply the second column by (-1) and add it to the third column:

$$\begin{vmatrix} 2 & 0 & 0 \\ -2 & 1 & 1 \\ 3 & -2 & 4 \end{vmatrix} \underset{(-1)\times C_2 + C_3 \to C_3}{=} \begin{vmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 6 \end{vmatrix} \underset{(property\ 10)}{=} 2\cdot 1\cdot 6 = 12.$$

The addition of elements of one column to the corresponding elements of other column can be also denoted with the help of arrows. The arrow points to the column where we write the obtained sums.

$$\begin{vmatrix} 2 & 0 & -2 \\ -2 & 1 & 3 \\ 3 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ -2 & 1 & 1 \\ 3 & -2 & 4 \end{vmatrix} \underset{\times(-1)}{=} \begin{vmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 6 \end{vmatrix} = 2\cdot 1\cdot 6 = 12.$$

<u>*Example 2.3.*</u> Evaluate the determinant

$$\begin{vmatrix} (a_1+b_1)^2 & (a_2+b_2)^2 & (a_3+b_3)^2 \\ a_1^2+b_1^2 & a_2^2+b_2^2 & a_3^2+b_3^2 \\ a_1b_1 & a_2b_2 & a_3b_3 \end{vmatrix}.$$

<u>*Solution.*</u>

$$\begin{vmatrix} (a_1+b_1)^2 & (a_2+b_2)^2 & (a_3+b_3)^2 \\ a_1^2+b_1^2 & a_2^2+b_2^2 & a_3^2+b_3^2 \\ a_1b_1 & a_2b_2 & a_3b_3 \end{vmatrix} =$$

$$= \begin{vmatrix} a_1^2+2a_1b_1+b_1^2 & a_2^2+2a_2b_2+b_2^2 & a_3^2+2a_3b_3+b_3^2 \\ a_1^2+b_1^2 & a_2^2+b_2^2 & a_3^2+b_3^2 \\ a_1b_1 & a_2b_2 & a_3b_3 \end{vmatrix} \times(-1) =$$

$$= \begin{vmatrix} 2a_1b_1 & 2a_2b_2 & 2a_3b_3 \\ a_1^2+b_1^2 & a_2^2+b_2^2 & a_3^2+b_3^2 \\ a_1b_1 & a_2b_2 & a_3b_3 \end{vmatrix} \begin{array}{c} 2\times R_3 = R_1 \\ = 0. \\ (property\ 6) \end{array}$$

§2.3. <u>The determinant of $n-$th order</u>

The method of computing 3×3 determinant cannot be modified to work with 4×4 or greater determinants.

The definition of the determinant of $n-$th order matrix, $n>2$, is defined in terms of the determinant of $(n-1)-$th order matrix. This definition requires two additional definitions.

<u>Definition</u>. Let A is $n\times n$ matrix. The i,j ***minor of*** A which is corresponded to entry $a_{i\,j}$ is the $(n-1)$- order determinant of matrix obtained from A by deleting the $i-th$ row and the $j-th$ column. The i,j minor of A is denoted by $M_{i\,j}$. Otherwise speaking, we delete the row and the column of matrix A, that contains the element $a_{i\,j}$, and then calculate the determinant of obtaining $(n-1)$- order matrix.

<u>Definition</u>. The i,j ***cofactor of a square matrix*** A**,** which is corresponded to entry $a_{i\,j}$, is the product of the minor $M_{i\,j}$ and the number $(-1)^{i+j}$:

$$C_{i\,j}=(-1)^{i+j}\cdot M_{i\,j}. \tag{2.4}$$

<u>Example 2.4.</u> Find the minor and the cofactor of the entry a_{23} of the matrix A, where $A=\begin{pmatrix} 4 & -5 & 1 \\ 0 & -2 & 3 \\ 10 & 9 & 6 \end{pmatrix}$.

<u>Solution.</u> The entry $a_{23}=3$ is located in the second row and the third column.

Removing all the elements of the second row and third column of A and forming the determinant of the remaining terms gives the minor M_{23}:

$$\begin{vmatrix} 4 & -5 & 1 \\ 0 & -2 & 3 \\ 10 & 9 & 6 \end{vmatrix} \qquad M_{23}=\begin{vmatrix} 4 & -5 \\ 10 & 9 \end{vmatrix}=4\cdot 9-10\cdot(-5)=86.$$

The cofactor of the entry a_{23} of the matrix A are evaluated from formula (2.4):

$$C_{23}=(-1)^{2+3}\cdot M_{23}=(-1)^{5}\cdot 86=-86.$$

<u>Theorem.</u> Let A is $n\times n$ matrix. Then the determinant of A may be found by choosing a row (or column) and summing the products of the entries of the chosen row (or column) and their cofactors:

$$det\,A=a_{i1}\cdot C_{i1}+a_{i2}\cdot C_{i2}+\cdots+a_{i\,n}\cdot C_{i\,n}, \tag{2.5}$$

$$det\,A=a_{1\,j}\cdot C_{1\,j}+a_{2\,j}\cdot C_{2\,j}+\cdots+a_{n\,j}\cdot C_{n\,j}. \tag{2.6}$$

Equations (2.5) and (2.6) are known as expansion along the $i-th$ row and expansion along the $j-th$ column respectively. Such the sums are called Laplace expansions.

Example 2.4. Evaluate the determinant of 4-th order $\Delta = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 1 \\ 1 & 0 & 5 & 3 \\ 6 & 4 & 1 & -1 \end{vmatrix}$.

Solution. We evaluate the determinant, by expanding along the second column (since it has two zeros).

Calculating cofactors with help of formula (2.4) we get

$$C_{12} = (-1)^{1+2} \cdot M_{12} = -\begin{vmatrix} 2 & 1 & 1 \\ 1 & 5 & 3 \\ 6 & 1 & -1 \end{vmatrix} = -(-10+18+1-30-6+1) = -(-26) = 26;$$

$$C_{42} = (-1)^{4+2} \cdot M_{42} = \begin{vmatrix} 1 & 3 & 4 \\ 2 & 1 & 1 \\ 1 & 5 & 3 \end{vmatrix} = 3+3+40-4-18-5 = 19.$$

Thus we write $\det A$ as the Laplace expansion

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 1 \\ 1 & 0 & 5 & 3 \\ 6 & 4 & 1 & -1 \end{vmatrix} = a_{12} \cdot C_{12} + a_{22} \cdot C_{22} + a_{32} \cdot C_{32} + a_{42} \cdot C_{42} =$$

$$= 2 \cdot 26 + 0 + 0 + 4 \cdot 19 = 52 + 76 = 128.$$

Note that we can expand along any row or column but it is the easiest to use a row or column that contains the greatest number of zeros. Moreover, before using the Laplace expansion we can use the determinant's properties to transform a determinant into one that contains a row or column with all entries zero except possibly one. The determinant can then be easily expanded by this row or column. An example illustrates the process the best.

Example 2.5. Evaluate the determinant $\begin{vmatrix} 3 & -1 & 2 \\ -2 & 4 & -3 \\ 4 & -2 & 5 \end{vmatrix}$.

Solution. It is convenient to work with the entry which is equal to 1 or -1. We will be create zeros in the second column.

$$\begin{vmatrix} 3 & -1 & 2 \\ -2 & 4 & -3 \\ 4 & -2 & 5 \end{vmatrix} \begin{matrix} \times 4 \\ \leftarrow \\ \end{matrix} = \begin{vmatrix} 3 & -1 & 2 \\ 10 & 0 & 5 \\ 4 & -2 & 5 \end{vmatrix} \begin{matrix} \times(-2) \\ \\ \leftarrow \end{matrix} = \begin{vmatrix} 3 & -1 & 2 \\ 10 & 0 & 5 \\ -2 & 0 & 1 \end{vmatrix} = -1 \cdot C_{12} =$$

$$=(-1)\cdot(-1)^{1+2}\cdot M_{12}=\begin{vmatrix}10 & 5\\ -2 & 1\end{vmatrix}=10\cdot 1-(-2)\cdot 5=20.$$

§2.4. The inverse of a matrix

Definition. Let A is $n\times n$ matrix. The ***inverse of the matrix*** A is a matrix A^{-1} such that $A\cdot A^{-1}=A^{-1}\cdot A=I$ where I is the $n\times n$ unit matrix.

Definition. If a matrix A has an inverse matrix, then the matrix A is called ***invertible.***

Note that not every matrix has an inverse.

Definition. A square matrix which determinant is zero is called a ***singular matrix***; otherwise it is ***nonsingular.***

Theorem. For each square nonsingular matrix A ($\det A\neq 0$) there exist a unique inverse matrix A^{-1}.

The inverse matrix A^{-1} is found by the formula

$$A^{-1}=\frac{1}{det\,A}(C)^{T}=\frac{1}{det\,A}adj\,A\ , \tag{2.7}$$

where the matrix C is the matrix of cofactors of the elements of A; the transpose of C is called the adjoint of A and is denoted by $adj\,A$.

Thus, to find the inverse matrix A^{-1}:

- Evaluate $\det A$ ($\det A\neq 0$).
- Construct the matrix C containing the cofactors of the elements of A .
- Take the transpose of the matrix of cofactors to find the adjoint $C^{T}=adj\,A$.
- multiply the matrix $adj\,A$ by the constant $\dfrac{1}{det\,A}$.

Example 2.6. Find the inverse of the matrix $A=\begin{pmatrix}3 & 4 & 1\\ 0 & 2 & 3\\ 5 & -1 & 6\end{pmatrix}$.

Solution. We first determine $det\,A$.

$$\det A=\begin{vmatrix}3 & 4 & 1\\ 0 & 2 & 3\\ 5 & -1 & 6\end{vmatrix}=36+60+0-10-0+9=95.$$

This is non-zero and so an inverse matrix can be constructed. To do this we calculate cofactor of the each entry of A:

$$C_{11}=(-1)^{1+1}\cdot\begin{vmatrix}2 & 3\\ -1 & 6\end{vmatrix}=12-(-3)=15;\quad C_{12}=(-1)^{1+2}\cdot\begin{vmatrix}0 & 3\\ 5 & 6\end{vmatrix}=-(0-15)=15;$$

$$C_{13} = (-1)^{1+3} \cdot \begin{vmatrix} 0 & 2 \\ 5 & -1 \end{vmatrix} = 0 - 10 = -10; \quad C_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 4 & 1 \\ -1 & 6 \end{vmatrix} = -\left(24 - (-1)\right) = -25;$$

$$C_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 3 & 1 \\ 5 & 6 \end{vmatrix} = 18 - 5 = 13; \qquad C_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 3 & 4 \\ 5 & -1 \end{vmatrix} = -(-3 - 20) = 23;$$

$$C_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} = 12 - 2 = 10; \qquad C_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 3 & 1 \\ 0 & 3 \end{vmatrix} = -(9 - 0) = -9;$$

$$C_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6.$$

This gives the matrix of cofactors as $C = \begin{pmatrix} 15 & 15 & -10 \\ -25 & 13 & 23 \\ 10 & -9 & 6 \end{pmatrix}$.

To find the adjoint , we take the transpose of the above, giving

$$C^T = adj\, A = \begin{pmatrix} 15 & -25 & 10 \\ 15 & 13 & -9 \\ -10 & 23 & 6 \end{pmatrix}.$$

Finally, we multiply the adjoint by the constant $\frac{1}{det\, A}$ to find the inverse giving

$$A^{-1} = \frac{1}{95} \cdot \begin{pmatrix} 15 & -25 & 10 \\ 15 & 13 & -9 \\ -10 & 23 & 6 \end{pmatrix} = \begin{pmatrix} \frac{15}{95} & \frac{-25}{95} & \frac{10}{95} \\ \frac{15}{95} & \frac{13}{95} & \frac{-9}{95} \\ \frac{-10}{95} & \frac{23}{95} & \frac{6}{95} \end{pmatrix}.$$

To check this answer, we need only find the product of matrices A and A^{-1} and make sure that its equal to identity matrix.

$$A\cdot A^{-1}=\begin{pmatrix}3&4&1\\0&2&3\\5&-1&6\end{pmatrix}\cdot\begin{pmatrix}\frac{15}{95}&\frac{-25}{95}&\frac{10}{95}\\\frac{15}{95}&\frac{13}{95}&\frac{-9}{95}\\\frac{-10}{95}&\frac{23}{95}&\frac{6}{95}\end{pmatrix}=$$

$$=\begin{pmatrix}\frac{45}{95}+\frac{60}{95}-\frac{10}{95}&-\frac{75}{95}+\frac{52}{95}+\frac{23}{95}&\frac{30}{95}-\frac{36}{95}+\frac{6}{95}\\0+\frac{30}{95}-\frac{30}{95}&0+\frac{26}{95}+\frac{69}{95}&0-\frac{18}{95}+\frac{18}{95}\\\frac{75}{95}-\frac{15}{95}-\frac{60}{95}&-\frac{125}{95}-\frac{13}{95}+\frac{138}{95}&\frac{50}{95}+\frac{9}{95}+\frac{36}{95}\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

Note that finding the inverse of a matrix A may be carried out in a number of ways.

We have already seen how to find the inverse of a matrix by using the matrix of cofactors. It is also possible to find the inverse by the Gauss Jordan procedure of elimination. This method will be discussed later in the next chapter.

Note that the method which is based on the formula (2.7) uses a very large number of operations (of the order of $n!$, where n is the dimension of the matrix), whereas elimination is only of the order n^3.

§2.5. The rank of a matrix

The rank of a general $m\times n$ matrix is an important concept, particularly in the computation of the number of solutions of a system of linear equations.

Like the determinant, the rank of matrix A is a single number that depends on the elements of A. Unlike the determinant, however, the rank of a matrix can be defined even when A is not square.

Definition. Let A be $m\times n$ matrix. A ***minor of*** A ***of*** k ***-th order*** is the determinant of some k-th order square matrix (called submatrix), cut down from A by removing one or more of its rows or columns.

We obtain the minors of order k from A by first deleting $m-k$ rows and $n-k$ columns, and then computing the determinant. There are usually many minors of A of a given order.

The definition of a minor of A can be formulated as follows.

Definition. Let A be $m\times n$ matrix. If to choose in this matrix arbitrarily k rows and k columns, where $k\le min(m,n)$ then elements standing on an intersection of these rows and columns, form a square matrix of the k-th order. The determinant of this matrix is called a ***minor of the*** k ***-th order*** of a matrix A.

Example 2.7. Find the minors of order 3 and 2 of the matrix

$$A=\begin{pmatrix}1 & 1 & -2 & 0\\ 2 & 0 & 2 & 2\\ 4 & 1 & 1 & 3\end{pmatrix}.$$

Solution. We obtain the determinants of order 3 by keeping all the rows and deleting one column from A. So there are four different minors of order 3. We compute one of them to illustrate (deleted the last column):

$$\begin{vmatrix}1 & 1 & -2\\ 2 & 0 & 2\\ 4 & 1 & 1\end{vmatrix}=0+8-4-0-2-2=0.$$

The minors of order 3 are the maximal minors of A, since there are no 4×4 submatrices of A. It may be proved that there are 18 minors of order 2 and 12 minors of order 1.

To find the second order minor it is necessary to delete any one row and two columns from A. For instance, ignoring the first and the third columns and the last row (it's the same as choose the first and the second rows and the second and fours column) we form 2×2 submatrix $\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$ and corresponding minor is $\begin{vmatrix}1 & 0\\ 0 & 2\end{vmatrix}=2$.

Definition. Let A be $m\times n$ matrix. The ***rank of*** A is the maximal order of a non-zero minor of A. It is denoted as *r(A)* or *rangA.*

Differently, the rank of a matrix *A* is equal *r*, if:

1) there is at least one minor of the *r*-th order of a matrix *A*, not equal to zero;

2) all minors of the (*r* +1)-th order are equal to zero or do not exist.

It is obvious, that the rank of rectangular matrix cannot be greater than the smallest of its sizes.

One of the methods of determining the rank of the matrix is based directly on the definition and it is called the method of covering minors.

Start with the minors of minimal order 1. If all minors of order 1 (i.e. all entries in A) are zero, then $rang\,A=0$. If there is one that is non-zero, then $rang\,A\geq 1$ and we continue with the minors of the second order (the entries of the previous minor must be included in the next). If there exist the second order nonzero minor, then $rang\,A\geq 2$ and we continue with the next large minors and so on, until we find that all next higher order minors are equal to zero.

Example 2.8. Determine the rank of the matrix $A=\begin{pmatrix}1 & 1 & -2 & 0\\ 2 & 0 & 2 & 2\\ 4 & 1 & 1 & 3\end{pmatrix}$.

Solution. Conside any the first order minor of A which is not equals to zero. For instance, choose the first row and the first column and form the minor

$|a_{11}| = |1| = 1 \neq 0$. The next bigger minor of A is the second order, so we choose 2×2 nonzero covering minor, for example, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2 \neq 0$.

There are four different minors of order 3, but we must evaluated only two of them which are covered the previous minor:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 0 & 2 \\ 4 & 1 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 4 & 1 & 3 \end{vmatrix} = 0.$$

If all the third covered minors are zero, then $rang\, A = 2$.

Another common approach to finding the rank of a matrix is to reduce it to a simpler form, generally Row Echelon Form(REF in short), by elementary row operations. This effective method of determining the rank is called Gauss elimination method.

Definition. The ***elementary row operations*** consist of the following:

1. Switch (interchange) two rows;
2. Multiply a row by a nonzero number;
3. Replace a row by the sum of that row and a multiple of another row.

The elementary row operations of matrix do not change the rank of this matrix.

Definition. When a new matrix is obtained from the original one by a sequence of elementary row operations, the new matrix is said to be **row-equivalent** to the original. The row-equivalent matrices are interconnected by a symbol $\sim$.

Definition. A matrix is in **Row Echelon Form** if

- all nonzero rows (rows with at least one nonzero element) are above any rows of all zeroes (all zero rows, if any, are at the bottom of the matrix);
- the leading coefficient (the first nonzero number from the left, also called the pivot) of a nonzero row is always strictly to the right of the leading coefficient of the row above it .

These two conditions mean that all entries in a column below a leading coefficient are zeros.

This is an example of matrices in row echelon form:

$$\begin{pmatrix} 3 & 5 & 0 & 7 & -1 \\ 0 & 0 & -2 & 5 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 2 \\ 0 & -2 & 5 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -5 & 3 & 0 & 1 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & 3 & 7 \end{pmatrix}.$$

To compute rank of a matrix through elementary row operations, simply perform a sequence of elementary row operations (as a result we get row-equivalent matrix) until the matrix reach the Row Echelon Form. Then the number of nonzero rows indicates the rank of the matrix.

Example 2.9. Determine the rank of the matrix $A=\begin{pmatrix}1 & 1 & -2 & 0\\ 2 & 0 & 2 & 2\\ 4 & 1 & 1 & 3\end{pmatrix}$ using Gauss elimination method (the matrix A is the same as the one given above in the example 2.8).

Solution. The first goal is to produce zeros below the first entry (the leading coefficient or pivot) in the first column.

$$\begin{pmatrix}1 & 1 & -2 & 0\\ 2 & 0 & 2 & 2\\ 4 & 1 & 1 & 3\end{pmatrix}\times(-2) : \begin{pmatrix}1 & 1 & -2 & 0\\ 0 & -2 & 6 & 2\\ 4 & 1 & 1 & 3\end{pmatrix}\times(-4) : \begin{pmatrix}1 & 1 & -2 & 0\\ 0 & -2 & 6 & 2\\ 0 & -3 & 9 & 3\end{pmatrix}$$

The second goal is to produce zeros below the second leading coefficient -2 in the second column. To accomplish this add $-3/2$ times the second row to the third row.

$$\begin{pmatrix}1 & 1 & -2 & 0\\ 0 & -2 & 6 & 2\\ 0 & -3 & 9 & 3\end{pmatrix}\times\left(-\tfrac{3}{2}\right) : \begin{pmatrix}1 & 1 & -2 & 0\\ 0 & -2 & 6 & 2\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Because the Row Echelon Form of the matrix A has two nonzero rows, we know that $rang\, A=2$. (The result is the same as in example 2.8).

Check yourself

1. Give the formula of evaluating determinant of the second order.
2. Explain a mnemonic Triangle's rule for for the 3 × 3 matrix determinant.
3. Explain a mnemonic Sarrus's rule for for the 3 × 3 matrix determinant.
4. Formulate the properties of determinant.
5. How to compute the determinant using cofactor expansion?
6. Explain the techniques that help compute the determinant easier. Give an example.
7. A square matrix can be singular or nonsingular. What does that mean?
8. Show $det(cA)=c^n\, det\, A$ where here A is an $n\times n$ matrix and c is a scalar.
9. How to find the inverse of a matrix by using cofactors?
10. If A^{-1} exist, what is the relationship between $det\, A$ and $det\, A^{-1}$? Explain your answer.
11. Suppose A is an upper triangular matrix. Show that A^{-1} exist if and only if all elements of the main diagonal are nonzero.
12. If A is an upper triangular matrix is it true that A^{-1} will also be upper triangular? Explain using the adjoint of A.
13. If the inverse of a square matrix exists, is it unique?
14. What types of operations are called elementary row operations?
15. When we say that matrix is in row echelon form?
16. Give the definitions of entry's minor M_{ij} and minor of rectangular matrix.

17. How many the third order minors can be formed from 3×5 matrix? How is changed the answer when the covering third order minors will be considered?

18. What is the rank of the identity matrix of the second order?

19. If a square matrix of n-th order is invertible what is the rank of this matrix?

20. For which value of x the rank of the matrix $\begin{pmatrix} -3 & 2 & -1 \\ 6 & 4 & x \\ 5 & -2 & 1 \end{pmatrix}$ is not equals to 3?

Problem set

1. Calculate the determinants:

a) $\begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix}$; b) $\begin{vmatrix} a+b & b \\ -b & a-b \end{vmatrix}$; c) $\begin{vmatrix} 1 & 3 & -7 \\ 4 & 2 & 1 \\ 3 & -1 & 8 \end{vmatrix}$; d) $\begin{vmatrix} 5 & 3 & 2 \\ -1 & 2 & 4 \\ 7 & 3 & 6 \end{vmatrix}$.

2. Calculate the determinants:

a) $\begin{vmatrix} 3 & -3 & -2 & -5 \\ 2 & 5 & 4 & 6 \\ 5 & 5 & 8 & 7 \\ 4 & 4 & 5 & 6 \end{vmatrix}$; b) $\begin{vmatrix} 3 & -5 & -2 & 2 \\ -4 & 7 & 4 & 4 \\ 4 & -9 & -3 & 7 \\ 2 & -6 & -3 & 2 \end{vmatrix}$.

3. Calculate the determinants, by expanding along the row or column, that consists only of letters:

a) $\begin{vmatrix} 2 & -3 & 4 & 1 \\ 4 & -2 & 3 & 2 \\ a & b & c & d \\ 3 & -1 & 4 & 3 \end{vmatrix}$; b) $\begin{vmatrix} 5 & a & 2 & -1 \\ 4 & b & 4 & -3 \\ 2 & c & 3 & -2 \\ 4 & d & 5 & -4 \end{vmatrix}$; c) $\begin{vmatrix} a & 3 & 0 & 5 \\ 0 & b & 0 & 2 \\ 1 & 2 & c & 3 \\ 0 & 0 & 0 & d \end{vmatrix}$.

4. Find the inverse of matrices:

a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$; b) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$; c) $\begin{pmatrix} 3 & -4 & 5 \\ 2 & -3 & 1 \\ 3 & -5 & -1 \end{pmatrix}$; d) $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$.

Note. Check the result by multiplication.

5. Given matrices: $A=\begin{pmatrix} 2 & 6 \\ -3 & 4 \end{pmatrix}$, $B=\begin{pmatrix} 3 & 0 \\ 2 & -5 \end{pmatrix}$.

Check the property of determinants $\det(A\cdot B)=\det A\cdot\det B$ for these matrices.

6. Given matrices: $A=\begin{pmatrix} 1 & 5 \\ -3 & -2 \end{pmatrix}$, $B=\begin{pmatrix} 3 & 0 \\ 2 & -5 \end{pmatrix}$. Verify the property $(AB)^{-1}=B^{-1}\cdot A^{-1}$ directly for these matrices.

7. Find the rank of the matrices in two different ways and verify equality:

a) $A=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{pmatrix}$; b) $A=\begin{pmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}$.

Answers

1. a) -2; b) a^2; c) 0; d) 68.
2. a) 90; b) 27.
3. a) $8a+15b+12c-19d$; c) $2a-8b+c+5d$; c) $abcd$.
5. Yes.
6. Yes.
7. a) *rang A=1*; b) *rang A=2*

Chapter 3. Systems of Linear Equations

§3.1. <u>The definition of a system of linear equations. Setup simultaneous linear equations in a matrix form. The solution of the linear system</u>

<u>*Definition*</u>. The system of m linear algebraic equations with n unknowns is the system of the following form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}, \tag{3.1}$$

where x_1, x_2,…, x_n are the unknowns; $a_{11}, a_{12}, \dots, a_{mn}$ are the coefficients of the system; numbers b_1, b_2,…, b_m are the constant terms.

<u>*Definition*</u>. The system (3.1) is called the ***homogeneous*** if all of the constant terms are zero, that is $b_1 = b_2 = \dots = b_m = 0$. If at least one of the numbers b_1, b_2, … , b_m is not equal to zero, then the system is called ***nonhomogeneous (or inhomogeneous).***

<u>*Definition*</u>. The matrix made up of the coefficients of the unknowns is called ***the coefficient matrix***:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \tag{3.2}$$

If write down the unknowns of the system and the constant terms as the column matrices X and B respectively

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \tag{3.3}$$

and use matrix multiplication then the system of equations (3.1) can be converted into following equivalent matrix form:

$$AX = B \tag{3.4}$$

<u>*Definition*</u>. ***A solution*** of system of linear equations (3.1) is an ordered list of n numbers $x_1^*, x_2^*, ..., x_n^*$, which, being substituted x_1^* for x_1, x_2^* for x_2, $\ldots$, x_n^* for x_n, will transform all the equations of the system into identities.

A linear system may behave in any one of three possible ways:

1. The system has infinitely many solutions.
2. The system has a single unique solution.
3. The system has no solution.

<u>*Definition*</u>. A system of equations is ***consistent*** if there is at least one solution of the system.

In contrast, an equation system is ***inconsistent*** if there is no set of values for the unknowns that satisfies all of the equations in other words a system hasn't got any solutions.

<u>*Definition*</u>. The consistent system that has only unique solutions, is called ***defined***. The consistent system, that has infinitely many solutions, is called ***undefined***.

A system with the same number of equations and unknowns has the square coefficient matrix. In the next two paragraphs we will consider the methods of solving of such systems.

§3.2. <u>Matrix solution</u>

The system of n linear equations with n unknowns can be written in the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n; \end{cases} \tag{3.5}$$

or in the matrix form (3.4), where the coefficient matrix of the system is a square matrix of order n:

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}; \tag{3.6}$$

the column of unknowns: $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$; (3.7)

the column of constant terms: $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$. (3.8)

Let the coefficient matrix A (3.6) is a nonsingular matrix (this means that $\det A \neq 0$, see §2.4.), then there exist a unique inverse matrix A^{-1}.

To find X from the matrix equation (3.4)

$$AX = B$$

we wish to 'get rid' of the A term on the left-hand side of equation, so we premultiply both sides of the equation (3.4) by A^{-1}:

$$A^{-1}(AX) = A^{-1}B.$$

Now using the associative property of matrix multiplication and $A^{-1}A = I$ where I is the $n \times n$ unit matrix, we have :

$$\left(A^{-1}A\right)X = A^{-1}B, \quad I\,X = A^{-1}B.$$

As the unit matrix multiplied by any matrix leaves it unchanged, we finally will get a formula for the solution of the matrix equation (3.4):

$$X = A^{-1}B \tag{3.9}$$

Note that the inverse method can be applied if and only if the coefficient matrix of a system is a square nonsingular matrix.

Example 3.1. Solve the system using matrix method: $\begin{cases} 2x_1 + x_2 - 3x_3 = -1; \\ x_1 \ - x_2 + 2x_3 = 1{,}5; \\ 3x_1 + 2x_2 \qquad = 3{,}5. \end{cases}$

Solution. Let's denote the matrices: $A = \begin{pmatrix} 2 & 1 & -3 \\ 1 & -1 & 2 \\ 3 & 2 & 0 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $B = \begin{pmatrix} -1 \\ 1{,}5 \\ 3{,}5 \end{pmatrix}$.

Now calculate the determinant of the matrix A

$$\det A = \begin{vmatrix} 2 & 1 & -3 \\ 1 & -1 & 2 \\ 3 & 2 & 0 \end{vmatrix} = 0 + 6 - 6 - 9 - 8 - 0 = -17.$$

The matrix A is nonsingular, therefore, let's find the inverse matrix A^{-1}.

$$A_{11}=+\begin{vmatrix}-1&2\\2&0\end{vmatrix}=-4,\qquad A_{12}=-\begin{vmatrix}1&2\\3&0\end{vmatrix}=6,\qquad A_{13}=+\begin{vmatrix}1&-1\\3&2\end{vmatrix}=5,$$

$$A_{21}=-\begin{vmatrix}1&-3\\2&0\end{vmatrix}=-6,\qquad A_{22}=+\begin{vmatrix}2&-3\\3&0\end{vmatrix}=9,\qquad A_{23}=-\begin{vmatrix}2&1\\3&2\end{vmatrix}=-1,$$

$$A_{31}=+\begin{vmatrix}1&-3\\-1&2\end{vmatrix}=-1,\qquad A_{32}=-\begin{vmatrix}2&-3\\1&2\end{vmatrix}=-7,\qquad A_{33}=+\begin{vmatrix}2&1\\1&-1\end{vmatrix}=-3.$$

Then according to the formula (2.7) for finding the inverse matrix one gets:

$$A^{-1}=\frac{1}{-17}\cdot\begin{pmatrix}-4&6&5\\-6&9&-1\\-1&-7&-3\end{pmatrix}^T=-\frac{1}{17}\cdot\begin{pmatrix}-4&-6&-1\\6&9&-7\\5&-1&-3\end{pmatrix}.$$

The solution of the system can be found by the formula (3.9):

$$X=A^{-1}\cdot B=-\frac{1}{17}\cdot\begin{pmatrix}-4&-6&-1\\6&9&-7\\5&-1&-3\end{pmatrix}\cdot\begin{pmatrix}-1\\1,5\\3,5\end{pmatrix}=-\frac{1}{17}\cdot\begin{pmatrix}4-9-3,5\\-6+13,5-24,5\\-5-1,5-10,5\end{pmatrix}=\begin{pmatrix}0,5\\1\\1\end{pmatrix}.$$

Since $X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$, one gets: $x_1=0,5;\quad x_2=1;\quad x_3=1.$

§3.3. Cramer's rule

Let's consider the system of two linear equations with two unknowns

$$\begin{cases}a_{11}x_1+a_{12}x_2=b_1,\\ a_{21}x_1+a_{22}x_2=b_2.\end{cases}\tag{3.10}$$

To solve the system, multiply the first equation by a_{22} and the second equation by $-a_{12}$, then add:

$$\begin{array}{rl} & a_{11}a_{22}x_1+a_{12}a_{22}x_2=\ b_1a_{22}\\ + & \\ & -a_{21}a_{12}x_1-a_{22}a_{12}x_2=-b_2a_{12}\\ \hline \end{array}$$

$x_1\left(a_{11}a_{22}-a_{21}a_{12}\right)=b_1a_{22}-b_2a_{12}$, from where

$$x_1=\frac{b_1a_{22}-b_2a_{12}}{a_{11}a_{22}-a_{21}a_{12}}.$$

Similarly, starting with the system (3.10) and eliminating x_1, we will get

$$x_2=\frac{b_2a_{11}-b_1a_{21}}{a_{11}a_{22}-a_{21}a_{12}}.$$

Taking into account the definition of the second order determinant (2.2) the formulas for x_1 and x_2 can be represented

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{\Delta_2}{\Delta},$$

where $\Delta = det\, A$ is the determinant of the coefficient matrix A of the system (3.10), Δ_1 is obtained from Δ by replacing the first column(the column of coefficients of x_1) by the column of constant terms, Δ_2 is obtained from Δ by replacing the second column(the column of coefficients of x_2) by the column of constant terms.

This result is called Cramer's Rule for the system of two equations with two unknowns. Now state the general form of Cramer's Rule.

Cramer's Rule. Let Δ is the determinant of the coefficient matrix A (3.6) of the system of n linear equations with n unknowns (3.5). If $\Delta \neq 0$, then the system has a unique solution given by the formulas:

$$x_i = \frac{\Delta_i}{\Delta} \qquad \text{for } i = 1, 2, \ldots, n \tag{3.11}$$

where Δ_i is the determinant obtained from Δ by replacing the column of coefficients of x_i by the column of constant terms.

Note, that in practice, Cramer's Rule is rarely used to solve systems of an order higher than 2 or 3.

<u>Example 3.2.</u> Solve the system using Cramer's method: $\begin{cases} 2x_1 + x_2 - 3x_3 = -1; \\ x_1 - x_2 + 2x_3 = 1,5; \\ 3x_1 + 2x_2 = 3,5. \end{cases}$

<u>Solution.</u> Let's denote the matrices: $A = \begin{pmatrix} 2 & 1 & -3 \\ 1 & -1 & 2 \\ 3 & 2 & 0 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $B = \begin{pmatrix} -1 \\ 1,5 \\ 3,5 \end{pmatrix}$

and

calculate $\Delta = \det A = \begin{vmatrix} 2 & 1 & -3 \\ 1 & -1 & 2 \\ 3 & 2 & 0 \end{vmatrix} = -17$ (see ex. 3.1).

Since $\Delta \neq 0$, then the solution of the system can be found using Cramer's rule.

By substituting at the first, and after that the second and third columns of determinant Δ on the column of constant terms, one gets three new determinants $\Delta_1, \Delta_2, \Delta_3$. Let's calculate them (check the solution as the exercise):

$$\Delta_1 = \begin{vmatrix} -1 & 1 & -3 \\ 1,5 & -1 & 2 \\ 3,5 & 2 & 0 \end{vmatrix} = -8,5; \qquad \Delta_2 = \begin{vmatrix} 2 & -1 & -3 \\ 1 & 1,5 & 2 \\ 3 & 3,5 & 0 \end{vmatrix} = -17; \qquad \Delta_3 = \begin{vmatrix} 2 & 1 & -1 \\ 1 & -1 & 1,5 \\ 3 & 2 & 3,5 \end{vmatrix} = -17.$$

By Cramer's formulas (3.11)

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{-8,5}{-17} = 0,5; \quad x_2 = \frac{\Delta_2}{\Delta} = \frac{-17}{-17} = 1; \quad x_3 = \frac{\Delta_3}{\Delta} = \frac{-17}{-17} = 1.$$

Therefore, the solution of system $x_1 = 0,5; \quad x_2 = 1; \quad x_3 = 1.$

§3.4. Gaussian elimination (the essence of the method)

Gaussian elimination (also known as row reduction) is the method that does not have the limitations in application and gives the solution of linear system using the less amount of operations, especially when we deal with the systems of large sizes.

Let's consider the system (3.1):

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases},$$

where $A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

are the coefficient matrix, the column matrix of the unknowns and the column matrix of the constant terms respectively.

Firstly explain the essence of the method. Among coefficients of the first unknown x_1 there is at least one which is not equal to 0. If $a_{11} = 0$ than we may write as a first equation the one where $a_{i1} \neq 0$. It means that we simply switch two equations of the system. So we can assume, without restricting the generality, that $a_{11} \neq 0$.

Now, with help of this first equation we can cancel or eliminate the first unknown x_1 from remaining $m-1$ equations. Really, if we multiply the first equation by the coefficient $\left(-\frac{a_{21}}{a_{11}}\right)$ and add to the second equation, we cancel x_1 from the second one.

In general, if we multiply the first equation by the coefficient $\left(-\frac{a_{i1}}{a_{11}}\right)$ and add to the *i*-th equation, we cancel x_1 from the *i*-th equation.

As a result we get the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \qquad a'_{22}x_2 + \cdots + a'_{2n}x_n = b'_2 \\ \qquad \qquad \cdots \\ \qquad a'_{m2}x_2 + \cdots + a'_{mn}x_n = b'_m \end{cases},$$

where the first equation includes n unknowns but the remaining $m-1$ equations contain only $n-1$ unknowns.

Now hold over the first equation and repeat the above process with the remaining $m-1$ equations in $n-1$ unknowns. We get the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ \qquad a'_{22}x_2 + a'_{13}x_3 + \cdots + a'_{2n}x_n = b'_2 \\ \qquad \qquad a''_{33}x_3 + \cdots + a''_{3n}x_n = b''_3 \\ \qquad \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \qquad \qquad a''_{m3}x_3 + \cdots + a''_{mn}x_n = b''_m \end{cases}.$$

In the process of performing Gauss elimination we can sometimes get the equation in which each coefficient of unknowns is zero.

If the constant term of this equation also equals 0 that is $0x_1 + 0x_2 + \cdots + 0x_n = 0$ than this equation for any choice $x_1,\ x_2, \ldots, x_n$ reduces to $0 = 0$ which is true. So this equation has no effect on the solution set of system and actually be dropped from the system. Therefore the number k of equations of the system becomes $k < m$.

If the constant term of this equation isn't equal to 0 that is $0x_1 + 0x_2 + \cdots + 0x_n = b$, $b \neq 0$, than this equation can never hold for any values of $x_1,\ x_2, \ldots, x_n$. Hence, the system has no solution and the process of solving the system must be stopped.

Each such cycle reduces the number of unknowns and the number of remaining equations and we can get to the system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ \qquad a'_{22}x_2 + a'_{13}x_3 + \cdots + a'_{2n}x_n = b'_2 \\ \qquad\qquad a''_{33}x_3 + \cdots + a''_{3n}x_n = b''_3 \\ \qquad\qquad\qquad \cdots\cdots\cdots\cdots\cdots \\ \qquad\qquad a_{kk}^{(k-1)}x_k + \cdots + a_{kn}^{(k-1)}x_n = b_k^{(k-1)} \end{cases}, \tag{3.12}$$

where $k \le m$.

There are two possibilities:

1) $k = n$

We got the triangular system that look like

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ \qquad a'_{22}x_2 + a'_{13}x_3 + \cdots + a'_{2n}x_n = b'_2 \\ \qquad\qquad a''_{33}x_3 + \cdots + a''_{3n}x_n = b''_3 \\ \qquad\qquad\qquad \cdots\cdots\cdots\cdots\cdots \\ \qquad\qquad\qquad a_{nn}^{(n-1)}x_n = b_n^{(n-1)} \end{cases} \tag{3.13}$$

and back substitution can be used to find the solution of the system. It means that the equations of the system (3.13) are solved starting from the last equation in one unknown x_n then substitute back into the $(n-1)$-th equation to obtain a solution for x_{n-1}, etc. In this case, the system has a single solution.

2) $k < n$

We got the system of a kind of trapeze (see 3.12) which has fewer equations than unknowns. From the last equation of the system (3.12) the leading unknown x_k can be expressed through unknowns $x_{k+1}, x_{k+2}, ..., x_n$ which are called free unknowns. Back substitution into the $(k-1)$-th equation gives a solution for x_{k-1} which is also expressed through free unknowns $x_{k+1}, x_{k+2}, ..., x_n$, etc. Since the free unknowns can be assigned any values, the system will have infinitely many solutions.

§3.5. <u>The Row Reduction Algorithm</u>

We can carry out reduction not only the system but the matrix which is corresponded to this system.

<u>*Definition.*</u> The matrix, which is composed of the coefficient matrix and the column of the constant terms is called the ***augmented matrix of the system*** and is denoted as $A|B$:

$$A|B=\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right). \tag{3.14}$$

The vertical line is included only to separate the coefficients of the unknowns from the constant terms. Each row of the augmented matrix gives the corresponding coefficients of an equation so we can perform the same operations on the row of an augmented matrix to reduce it as we do above on equations of a linear system. The three elementary row operations we can use to obtain the row-equivalent matrix. Recall them (see §2.5.):

4. Switch (interchange) two rows;
5. Multiply a row by a nonzero number;
6. Replace a row by the sum of that row and a multiple of another row.

The theoretical foundation of Gaussian elimination is the next statement: if two matrices are row-equivalent augmented matrices of two systems of linear equations, then the two systems have the same solution sets – a solution of the one system is the solution of the other.

Solving the system of equations by Gaussian elimination can be divided into two parts and can be summarized as follows.

First write down the augmented matrix $A|B$, then perform a sequence of elementary row operations to transform – or reduce – the original augmented matrix $A|B$ into one of the form $A_1|B_1$ where A_1 is in Row Echelon Form (see §2.5.) (the first part which is called ***forward*** elimination). The solutions of the system represented by the simpler augmented matrix $A_1|B_1$ can be found by inspection of the bottom nonzero rows and back-substitution into the higher rows (the second part which is called ***backward*** elimination).

Recall that a matrix $A_1=\left(a^1_{ij}\right)$ is in Row Echelon Form when $a^1_{ij}=0$ for $i>j$, any zero rows appear at the bottom of the matrix, and the first nonzero entry in any row is to the right of the first nonzero entry in any higher row.

Example 3.3. Solve the following system using Gaussian elimination:

$$\begin{cases} x_1+2x_2-x_3=-1, \\ 2x_1-3x_2+x_3=4, \\ 3x_1+x_2+4x_3=11. \end{cases}$$

Solution. The augmented matrix which represents this system is

$$A|B=\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & -3 & 1 & 4 \\ 3 & 1 & 4 & 11 \end{array}\right).$$

The first aim is to produce zeros below the first entry in the first column, which interpret into eliminating the first unknowns x_1 from the second and third equations:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & -3 & 1 & 4 \\ 3 & 1 & 4 & 11 \end{array}\right)\begin{matrix}\times(-2)\\ \\ \\ \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -7 & 3 & 6 \\ 3 & 1 & 4 & 11 \end{array}\right)\begin{matrix}\times(-3)\\ \\ \\ \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -7 & 3 & 6 \\ 0 & -5 & 7 & 14 \end{array}\right)$$

The second aim is to produce zeros below the second entry in the second column, which interpret into eliminating the second unknowns x_2 from the third equations. One way to accomplish this would be to add $-5/7$ times the second row to the third row. However, in order to avoid fractions, firstly create 1 as the leading coefficient (the pivot) of the second row:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -7 & 3 & 6 \\ 0 & -5 & 7 & 14 \end{array}\right)\begin{matrix} \\ \times 2\\ \times 3\end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -14 & 6 & 12 \\ 0 & -15 & 21 & 42 \end{array}\right)\begin{matrix} \\ \\ \times(-1)\end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & -15 & -30 \\ 0 & -15 & 21 & 42 \end{array}\right)\begin{matrix} \\ \\ \times 1/3\end{matrix} \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & -15 & -30 \\ 0 & -5 & 7 & 14 \end{array}\right)$$

Now, add 5 times the second row to the third row:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & -15 & -30 \\ 0 & -5 & 7 & 14 \end{array}\right)\begin{matrix} \\ \times 5\\ \\ \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & -15 & -30 \\ 0 & 0 & -68 & -136 \end{array}\right)\begin{matrix} \\ \\ \times(-1/68)\end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & -15 & -30 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Since the coefficient matrix has been transformed into Row Echelon Form, the forward elimination part is complete.

Let's consider backward elimination part. The system, which is corresponded to the last augmented matrix, is

$$\begin{cases} x_1 + 2x_2 - x_3 = -1, \\ \quad x_2 - 15x_3 = -30, \\ \qquad\quad x_3 = 2. \end{cases}$$

Let's find x_2 from the second equation: $x_2 = -30 + 15x_3$. Back-substitution of the value $x_3 = 2$ into the second equation gives $x_2 = -30 + 15 \cdot 2 = 0$.

Let's determine x_1 from the first equation: $x_1 = -1 + 2x_2 + x_3$. Back-substitution of both values $x_3 = 2$ and $x_2 = 0$ into the first equation gives $x_1 = -1 - 2 \cdot 0 + 2 = 1$.

Therefore, the solution of system: $x_1=1$, $x_2=0$, $x_3=2$.

§3.6. Gauss−Jordan reduction

Gauss−Jordan reduction is a continuation of the forward elimination of Gauss elimination and a transformation of the augmented matrix from the Row Echelon Form to the Reduced Row Echelon Form.

Definition. A matrix is in **Reduced Row Echelon Form** (olso called Row Canonical Form) if

- the matrix is in Row Echelon Form;
- the each leading coefficient is equal to 1 and is the only nonzero entry in its column.

Unlike the Row Echelon Form, the Reduced Row Echelon Form of a matrix is unique and does not depend on the algorithm used to compute it.

Briefly speaking, Gaussian elimination works from the top down, to transform an augmented matrix into in a Row Echelon Form, whereas Gauss−Jordan elimination continues where Gaussian forward elimination part left off by then works from the bottom up to produce a matrix in the Reduced Row Echelon Form.

Let's illustrated Gauss−Jordan reduction in the following example.

Example 3.4. Solve the following system using Gauss−Jordan elimination:

$$\begin{cases} x_1 + 2x_2 - x_3 = -1, \\ 2x_1 - 3x_2 + x_3 = 4, \\ 3x_1 + x_2 + 4x_3 = 11. \end{cases}$$

Solution. The previous example 3.3 shows how forward elimination part reduces the original augmented matrix. Let's write down the transformed augmented matrix of example 3.3 and let's continue to perform elementary row operations to get zeroes above the main diagonal:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & -15 & -30 \\ 0 & 0 & 1 & 2 \end{array}\right)\times 15 : \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right)\times 1 : \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right)\times(-2) : \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

The last coefficient matrix is in the Reduced Row Echelon Form and the final augmented matrix immediately gives the solution: $x_1=1$, $x_2=0$, $x_3=2$.

§3.7. Arbitrary systems of linear equations. Rouché-Capelli Theorem

Let's consider the system of m linear equations with n unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}. \tag{3.15}$$

In the general case, before we can solve the system of linear equations (3.15), it is important to know whether its solutions exist, or, in other words, if it is consistent. The answer on this question can give the **theorem of Rouché-Capelli:**

The system of linear equations (3.15) is consistent if and only if the rank of coefficient matrix is equal to the rank of augmented matrix of the system:

$$\boxed{\textbf{\textit{rang A = rang A}}\big|\textbf{\textit{B}} \quad \Leftrightarrow \quad \textbf{\textit{the system is consistent}}} \tag{3.16}$$

Suppose the ranks of two matrices are equal to the number r. This means that there exists the minor of rth order in the coefficient matrix of the system that is not equal to zero.

<u>*Definition.*</u> The non-zero minor of coefficient matrix of the system (3.16), the order of which is equal to the rank of this matrix, is called the ***leading minor.***

Let's assume that this is the minor that is composed of coefficients of first r unknowns. Let's leave the terms with these unknowns in the left hand side of the equations, and let's move the rest of terms to the right hand side. Let's discard all equations of the system (15.1) after rth one. Then the system (3.15) will take the following form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1r}x_r = b_1 - a_{1,r+1}x_{r+1} - \ldots - a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2r}x_r = b_2 - a_{2,r+1}x_{r+1} - \ldots - a_{2n}x_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rr}x_r = b_r - a_{r,r+1}x_{r+1} - \ldots - a_{rn}x_n \end{cases} \tag{3.17}$$

The unknowns x_1, x_2, ..., x_r , which coefficients form the leading minor, are called the ***leading*** or ***base***, and x_{r+1}, x_{r+2}, ..., x_n are the ***free*** unknowns.

The system (3.17) can be solved with respect to the leading unknowns x_1, x_2, ..., x_r, by employing the one of the described above ways, and assuming, that $\begin{pmatrix} b_1 - a_{1,r+1}x_{r+1} - \ldots - a_{1n}x_n \\ b_2 - a_{2,r+1}x_{r+1} - \ldots - a_{2n}x_n \\ \ldots \\ b_r - a_{r,r+1}x_{r+1} - \ldots - a_{rn}x_n \end{pmatrix}$ is the right hand side. In this case each of the leading unknowns x_1, x_2, ..., x_r can be expressed through free unknowns x_{r+1}, x_{r+2}, ..., x_n. If the specific numeric values are not assigned to the free unknowns, we have the so-called ***general solution*** of the system of equations (3.15). By assigning some numbers to free variables, we will get the ***particular solutions*** of this system. It is obvious, that there is the infinite number of particular solutions. Therefore, the consistent system, the rank of which is less than the number of unknowns is **underdetermined**.

Example 3.5. Solve the system $\begin{cases} x_1 + 2x_2 - x_3 + 4x_4 = 3, \\ 2x_1 - 3x_2 + x_3 \qquad = 0. \end{cases}$

Solution. The coefficient matrix of the system: $A = \begin{pmatrix} 1 & 2 & -1 & 4 \\ 2 & -3 & 1 & 0 \end{pmatrix}$, its rank cannot be greater than two. Let's calculate the minor of this matrix, taking, for instance, first two columns: $\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = -7$. Since this minor of 2nd order is not equal to zero, it is a leading minor, and unknowns x_1 and x_2 are leading unknowns. Let's move the terms with free unknowns x_3 and x_4 to right hand side of equations: $\begin{cases} x_1 + 2x_2 = 3 + x_3 - 4x_4, \\ 2x_1 - 3x_2 = -x_3. \end{cases}$

We have obtained the system, from which we can determine two leading unknowns x_1, x_2. The determinant of this system $\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix}$ is not equal to zero. Let's solve it in three ways.

I way. Let's apply Cramer's method:

$$\Delta = \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = -7;$$

$$\Delta_1 = \begin{vmatrix} 3 + x_3 - 4x_4 & 2 \\ -x_3 & -3 \end{vmatrix} = -3(3 + x_3 - 4x_4) + 2x_3 = -x_3 + 12x_4 - 9;$$

$$\Delta = \begin{vmatrix} 1 & 3 + x_3 - 4x_4 \\ 2 & -x_3 \end{vmatrix} = -x_3 - 2\left(3 + x_3 - 4x_4\right) = -3x_3 + 8x_4 - 6.$$

Thus, $x_1 = \frac{\Delta_1}{\Delta} = \frac{-x_3 + 12x_4 - 9}{-7} = \frac{1}{7}x_3 - \frac{12}{7}x_4 + \frac{9}{7}$, $x_2 = \frac{\Delta_2}{\Delta} = \frac{-3x_3 + 8x_4 - 6}{-7} = \frac{3}{7}x_3 - \frac{8}{7}x_4 + \frac{6}{7}$.
The free unknowns can take any number values. Therefore, let's take: $x_3 = C_1$, $x_4 = C_2$. Then the general solution of the system is:

$$x_1 = \frac{1}{7}C_1 - \frac{12}{7}C_2 + \frac{9}{7}, \quad x_2 = \frac{3}{7}C_1 - \frac{8}{7}C_2 + \frac{6}{7}, \quad x_3 = C_1, \quad x_4 = C_2.$$

II way. Let's apply matrix method: the coefficient matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix}$, the right hand side $B = \begin{pmatrix} 3 + x_3 - 4x_4 \\ -x_3 \end{pmatrix}$, the matrix of unknowns $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Let's calculate: $\det A = -7$; the cofactors: $A_{11} = -3$, $A_{12} = -2$, $A_{21} = -2$, $A_{22} = 1$. Using formula (2.7) we find A^{-1}:

$$A^{-1} = \frac{1}{-7} \cdot \begin{pmatrix} -3 & -2 \\ -2 & 1 \end{pmatrix}^T = \begin{pmatrix} \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{pmatrix}.$$

The solution of the system can be found by the formula (3.9):

$$X=\begin{pmatrix}\frac{3}{7} & \frac{2}{7}\\ \frac{2}{7} & -\frac{1}{7}\end{pmatrix}\cdot\begin{pmatrix}3+x_3-4x_4\\ -x_3\end{pmatrix}=\begin{pmatrix}\frac{3}{7}\cdot(3+x_3-4x_4)-\frac{2}{7}x_3\\ \frac{2}{7}\cdot(3+x_3-4x_4)+\frac{1}{7}x_3\end{pmatrix}=\begin{pmatrix}\frac{1}{7}x_3-\frac{12}{7}x_4+\frac{9}{7}\\ \frac{3}{7}x_3-\frac{8}{7}x_4+\frac{6}{7}\end{pmatrix}.$$

This means, that the leading unknowns are expressed through free unknowns in the following manner: $x_1=\frac{1}{7}x_3-\frac{12}{7}x_4+\frac{9}{7}$, $x_2=\frac{3}{7}x_3-\frac{8}{7}x_4+\frac{6}{7}$. Taking $x_3=C_1$, $x_4=C_2$, we will get the general solution of the system:

$$x_1=\frac{1}{7}C_1-\frac{12}{7}C_2+\frac{9}{7},\quad x_2=\frac{3}{7}C_1-\frac{8}{7}C_2+\frac{6}{7},\quad x_3=C_1,\quad x_4=C_2.$$

<u>III way.</u> Let's apply Gauss–Jordan reduction. For this let's write down the given augmented matrix of the system and perform elementary row transformations to get identity matrix instead of leading minor:

$$A|B=\left(\begin{array}{cccc|c}1 & 2 & -1 & 4 & 3\\ 2 & -3 & 1 & 0 & 0\end{array}\right).$$

$$\left(\begin{array}{cccc|c}1 & 2 & -1 & 4 & 3\\ 2 & -3 & 1 & 0 & 0\end{array}\right)\times(-2) \sim \left(\begin{array}{cccc|c}1 & 2 & -1 & 4 & 3\\ 0 & -7 & 3 & -8 & -6\end{array}\right):(-7) \sim \left(\begin{array}{cccc|c}1 & 2 & -1 & 4 & 3\\ 0 & 1 & -3/7 & 8/7 & 6/7\end{array}\right)\times(-2) \sim$$

$$\sim\left(\begin{array}{cccc|c}1 & 0 & -1/7 & 12/7 & 9/7\\ 0 & 1 & -3/7 & 8/7 & 6/7\end{array}\right).$$

The last matrix corresponds to the system: $\begin{cases}x_1-1/7\,x_3+12/7\,x_4=9/7\\ x_2-3/7\,x_3+8/7\,x_4=6/7\end{cases}$, and let's express the leading variables from this matrix: $x_1=\frac{1}{7}x_3-\frac{12}{7}x_4+\frac{9}{7}$, $x_2=\frac{3}{7}x_3-\frac{8}{7}x_4+\frac{6}{7}$. Therefore, the general solution is: $x_1=\frac{1}{7}C_1-\frac{12}{7}C_2+\frac{9}{7}$, $x_2=\frac{3}{7}C_1-\frac{8}{7}C_2+\frac{6}{7}$, $x_3=C_1$, $x_4=C_2$.

<u>§3.8. Homogenous systems of equations</u>

Let's consider the homogenous system of *m* equations with *n* unknowns:

$$\begin{cases}a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=0\\ a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n=0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n=0\end{cases}\qquad(3.18)$$

The augmented matrix of the system (3.18) has the following form:

$$A|B=\left(\begin{array}{cccc|c}a_{11} & a_{12} & \cdots & a_{1n} & 0\\ a_{21} & a_{22} & \cdots & a_{2n} & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0\end{array}\right).$$

It is clear that the $rang\ A = rang\ A|B$. Thus, according to the theorem of Roché-Capelli, the ***homogenous system is always consistent***.

It should be noted that the set of unknown values

$$x_1 = 0,\ x_2 = 0,\ ...,\ x_n = 0 \tag{3.19}$$

transforms the equations of the system (3.18) into identities, therefore, (3.19) is its solution. The solution (3.19) of homogenous system is called the ***trivial solution***.

Suppose the rank of system (3.18) is equal to *r*. Then two cases are possible:

1) if *r* = *n*, then the system has only one ***trivial*** solution;
2) if *r* < *n*, then the system has infinite number of solutions. In this case the trivial solution is a special case of the set of solutions.

Suppose the rank *r* of matrix of linear equations system (3.18) is less than the number *n* of unknowns. This means, that the matrix *A* has the minor of *r*th order, that is not equal to zero. According to the general theory (§3.7) one can determine *r* leading unknowns and, respectively, *n–r* free unknowns. The system of equations (3.18) can be rewritten in the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1r}x_r = -a_{1,r+1}x_{r+1} - \ldots - a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2r}x_r = -a_{2,r+1}x_{r+1} - \ldots - a_{2n}x_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rr}x_r = -a_{r,r+1}x_{r+1} - \ldots - a_{rn}x_n \end{cases} \tag{3.19}$$

The system in the form of (3.19) can be solved by one of the following ways, described in §§3.3–3.6, as it is shown in example 3.5 (§3.5).

As a result of the procedure, one obtains *n* solutions of the system of equations (3.18) in the form of ordered sequence $x_1, x_2, \ldots, x_r, c_{r+1}, c_{r+2}, \ldots, c_n$, where the base variables $x_1, x_2, \ldots, x_r$ are expressed via *n–r* constants $c_{r+1}, c_{r+2}, \ldots, c_n$. Setting the constants $c_{r+1}, c_{r+2}, \ldots, c_n$ equal to the values:

$$\begin{array}{llll} c_{r+1} = 1, & c_{r+2} = 0, & ..., & c_n = 0; \\ c_{r+1} = 0, & c_{r+2} = 1, & ..., & c_n = 0; \\ \quad ... & \quad ... & ... & \quad ... \\ c_{r+1} = 0, & c_{r+2} = 0, & ..., & c_n = 1, \end{array} \tag{3.20}$$

one gets the **fundamental system of solutions** of homogenous system.

Let the number of equations of system (3.18) is equal to the number of unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0, \\ \ldots\ \ \ldots\ \ \ldots\ \ \ldots\ \ \ldots\ \ \ldots\ \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = 0. \end{cases} \tag{3.21}$$

The following statement can be formulated for the system (3.21):

If the determinant of the coefficient matrix $\Delta(A) \neq 0$, then the system (3.21) has only one trivial solution. If $\Delta(A) = 0$, then the system (3.21) has infinite number of solutions.

<u>Example 3.6.</u> Solve the system: $\begin{cases} 2x_1 + 4x_2 - x_3 + x_4 = 0, \\ 3x_1 - x_2 + x_3 - 2x_4 = 0, \\ x_1 + x_2 - 2x_3 + 3x_4 = 0. \end{cases}$

<u>Solution.</u> The system has 4 unknowns, and its rank cannot be greater than three. Therefore, this system has infinite number of solutions. Let's write down the augmented matrix of the system and let's perform the elementary rows operations (recall, that the elementary row operations change neither the rank of the system nor its solutions). Note, that the column of constant terms remains zero, that's why it can be omitted.

$$\begin{pmatrix} 2 & 4 & -1 & 1 \\ 3 & -1 & 1 & -2 \\ 1 & 1 & -2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -2 & 3 \\ 3 & -1 & 1 & -2 \\ 2 & 4 & -1 & 1 \end{pmatrix} \begin{array}{l} \times(-3)\ \times(-2) \\ \\ \\ \end{array} \sim \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & -4 & 7 & -11 \\ 0 & 2 & 3 & -5 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & 2 & 3 & -5 \\ 0 & -4 & 7 & -11 \end{pmatrix} \begin{array}{l} \\ \times 2 \\ \\ \end{array} \sim \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & 2 & 3 & -5 \\ 0 & 0 & 13 & -21 \end{pmatrix} \begin{array}{l} \\ :2 \\ :13 \end{array} \sim \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & 3/2 & -5/2 \\ 0 & 0 & 1 & -21/13 \end{pmatrix} \begin{array}{l} \\ \\ \times(-3/2)\ \times 2 \end{array} \sim$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & -3/13 \\ 0 & 1 & 0 & -1/13 \\ 0 & 0 & 1 & -21/13 \end{pmatrix} \begin{array}{l} \\ \times(-1) \\ \\ \end{array} \sim \begin{pmatrix} 1 & 0 & 0 & -2/13 \\ 0 & 1 & 0 & -1/13 \\ 0 & 0 & 1 & -21/13 \end{pmatrix}$$

The first three columns form the leading minor. Let's write down the system that corresponds to the last matrix and let's express the leading unknowns x_1, x_2, x_3 through the free unknown $x_4: x_1 = \frac{2}{13}x_4,\ \ x_2 = \frac{1}{13}x_4,\ \ x_3 = \frac{21}{13}x_4$. Setting $x_4 = C$, one gets the set of solutions:

$X = \begin{pmatrix} \frac{2}{13}C \\ \frac{1}{13}C \\ \frac{21}{13}C \\ C \end{pmatrix}$. When $C = 0$ we get the trivial solution: $X_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Example 3.7. Let's solve the system: $\begin{cases} 2x_1 + x_2 - x_3 = 0, \\ 3x_1 - x_2 + 6x_3 = 0, \\ x_1 + x_2 - 12x_3 = 0. \end{cases}$

Solution. The system has three equations and three unknowns, therefore, let's calculate the determinant:

$\begin{vmatrix} 2 & 1 & -1 \\ 3 & -1 & 6 \\ 1 & 1 & -12 \end{vmatrix} = 24 + 6 - 3 - 1 + 36 - 12 = 50 \neq 0$. Since the determinant is not equal to zero, then the system has only one trivial solution: $x_1 = 0,\ x_2 = 0,\ x_3 = 0.$

Check yourself

1. Give the definition of a solution of system of linear equations.
2. What conditions must satisfy the system in order to be solved by using the inverse of the coefficient matrix?
3. Formulate the Cramer's Rule for the system of linear equations. What the limitations in application has this method?
4. A system of equations can be consistent or inconsistent, defined or undefined. What does that mean?
5. How to find the solution of the system by using the inverse of the coefficient matrix?
6. What types of operations are used in Gauss elimination method?
7. Formulate Rouché-Capelli theorem.
8. Give the definition of a leading minor of coefficient matrix of system. How to choose base and free unknowns with help of this concept.
9. What the set of numbers is a solution of any homogenous system?
10. Write down a 2 by 2 system with infinitely many solutions.
11. Write down a 3 by 3 homogenous system with only one trivial solutions.
12. For which values of *m* does the system

$$\begin{cases} mx_1 + x_2 = 2, \\ x_1 + mx_2 = 2. \end{cases}$$

have no solution, one solution, or infinitely many solutions?

13. For two equations and two unknowns, the equations represent two lines. Give the geometrical interpretations of three possibilities of system's solution: there is a unique solution; there is no solution; there is an infinite number of solutions.
14. True or false: a system with more unknowns than equations has at least one solution. (As always, to say 'true' you must prove it, while to say 'false' you must produce a counterexample.)
15. For what values of a and b is the following system consistent?

$$\begin{cases} 3x_1 - 2x_2 = a, \\ -9x_1 + 6x_2 = b. \end{cases}$$

Problem set

1. Solve the matrix equation $AX = B,$ where

$$A=\begin{pmatrix}3 & 2\\ 4 & 3\end{pmatrix};\ X=\begin{pmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix},\ B=\begin{pmatrix}-1 & 7\\ 3 & 5\end{pmatrix}.$$

2. Solve the matrix equations:

a) $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}X=\begin{pmatrix}3 & 5\\ 5 & 9\end{pmatrix}$; b) $\begin{pmatrix}1 & 2 & -3\\ 3 & 2 & -4\\ 2 & -1 & 0\end{pmatrix}X=\begin{pmatrix}1 & -3 & 0\\ 10 & 2 & 7\\ 10 & 7 & 8\end{pmatrix}$.

3. Solve the system of equations by using: a) matrix method; b) Cramer's formulas; c) Gaussian elimination; d) Gauss-Jordan reduction.

$$\begin{cases}3x_1 - x_2 = 1\\ -2x_1 + x_2 + x_3 = -1\\ 2x_2 - x_2 + 4x_3 = -4\end{cases}$$

4. Solve the system of equations:

a) $\begin{cases}3x_1 - 2x_2 - 5x_3 + x_4 = 3\\ 2x_1 - 3x_2 + x_3 + 5x_4 = -3\\ x_1 + 2x_2 - 4x_4 = -3\\ x_1 - x_2 - 4x_3 + 9x_4 = 22\end{cases}$ b) $\begin{cases}4x_1 - 3x_2 + x_3 + 5x_4 = 7\\ x_1 - 2x_2 - 2x_3 - 3x_4 = 3\\ 3x_1 - x_2 + 2x_3 = -1\\ 2x_1 + 3x_2 + 2x_3 - 8x_4 = -7\end{cases}$

c) $\begin{cases}x_1 - 2x_2 + 5x_3 = -2\\ 2x_1 - 3x_2 + 4x_3 = -8\\ 4x_1 + x_2 - 3x_4 = -13\end{cases}$ d) $\begin{cases}-4x_1 + x_2 - 5x_3 = -8\\ 2x_1 + 2x_2 + 3x_3 = 7\\ -2x_1 + 2x_2 + 5x_4 = 6\end{cases}$

5. Solve the system using the theorem of Roché-Capelli:

a) $\begin{cases}3x_1 - 2x_2 + 5x_3 + 4x_4 = 2\\ 6x_1 - 4x_2 + 4x_3 + 3x_4 = 3\\ 9x_1 - 6x_2 + 3x_3 + 2x_4 = 4\end{cases}$ b) $\begin{cases}x_1 + x_2 + 3x_3 - 2x_4 + 3x_5 = 1\\ 2x_1 + 2x_2 + 4x_3 - x_4 + 3x_5 = 2\\ 3x_1 + 3x_2 + 5x_3 - 2x_4 + 3x_5 = 1\\ 2x_1 + 2x_2 + 8x_3 - 3x_4 + 9x_5 = 2\end{cases}$ c) $\begin{cases}9x_1 - 3x_2 + 5x_3 + 6x_4 = 4\\ 6x_1 - 2x_2 + 3x_3 + x_4 = 5\\ 3x_1 - x_2 + 3x_3 + 14x_4 = -8\end{cases}$

6. Solve the system using the Gauss-Jordan reduction:

a) $\begin{cases}2x_1 + 7x_2 + 3x_3 + x_4 = 6\\ 3x_1 + 5x_2 + 2x_3 + 2x_4 = 4\\ 9x_1 + 4x_2 + x_3 + 7x_4 = 2\end{cases}$ b) $\begin{cases}2x_1 - 3x_2 + 5x_3 + 7x_4 = 1\\ 4x_1 - 6x_2 + 2x_3 + 3x_4 = 2\\ 2x_1 - 3x_2 - 11x_3 - 15x_4 = 1\end{cases}$

7. Find the fundamental system of solutions of the system: $\begin{cases}x_1 + 2x_2 + 4x_3 - 3x_4 = 0\\ 3x_1 + 5x_2 + 6x_3 - 4x_4 = 0\\ 4x_1 + 5x_2 - 2x_3 + 3x_4 = 0\\ 3x_1 + 8x_2 + 24x_3 - 19x_4 = 0\end{cases}$

.

Answers

1. $X = \begin{pmatrix} -9 & 11 \\ 13 & -13 \end{pmatrix}$.

2. a) $X = \begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix}$ b) $X = \begin{pmatrix} 6 & 4 & 5 \\ 2 & 1 & 2 \\ 3 & 3 & 3 \end{pmatrix}$.

3. $x_1 = 1$, $x_2 = 2$, $x_3 = -1$.

4. a) $x_1 = -1$, $x_2 = 3$, $x_3 = -2$, $x_4 = 2$; b) $x_1 = 2$, $x_2 = 1$, $x_3 = -3$, $x_4 = 1$;
c) $x_1 = -3$, $x_2 = 2$, $x_3 = 1$; d) $x_1 = 1$, $x_2 = 1$, $x_3 = 1$.

5. a) $x_3 = 6 - 15x_1 + 10x_2; x_4 = -7 + 18x_1 - 2x_2;$ b) the system is inconsistent;

c) $x_3 = 2 - \frac{27}{13}x_1 + \frac{9}{13}x_2$, $\quad x_4 = -1 + \frac{3}{13}x_1 - \frac{1}{13}x_2$.

6. a) $x_1 = \frac{1}{11}(x_3 - 9x_4 - 2)$; $x_2 = \frac{1}{11}(-5x_1 + x_4 + 10)$; b) $x_3 = 22x_1 - 33x_2 - 11$;
$x_4 = -16x_1 + 24x_2 + 8$.

7. The general solution: $X = \begin{pmatrix} 8x_3 - 7x_4 \\ -6x_3 + 5x_4 \\ x_3 \\ x_4 \end{pmatrix}$; the fundamental system: $\begin{pmatrix} 8 \\ -6 \\ 1 \\ 0 \end{pmatrix}$; $\begin{pmatrix} -7 \\ 5 \\ 0 \\ 1 \end{pmatrix}$

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