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CONSTRUCTION OF STATIC SOLUTIONS OF THE EQUATIONS OF ELASTICITY AND THERMOELASTICITY THEORY

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Summary. *New solutions to the theories of thermoelasticity and elasticity in the Cartesian coordinate system are found in this paper. New explicit partial solutions of thermoelasticity equations, when the temperature field is defined by 3D or 2D harmonic functions, are constructed. Displacements, deformations, and stresses determined by these partial solutions are called temperature functions. A simple formula for the expression of normal temperature stresses is obtained and it is shown that their sum is zero. Separate cases when the temperature depends on the product of harmonic functions of two variables on the degree of coordinate z are also considered. Partial and general solutions are derived for them. General solutions of thermoelasticity equations (Navier's equations) through four harmonic functions, when the temperature field is given three-dimensional or two-dimensional harmonic functions, are constructed. The thermoelastic state of the body is divided into symmetric and asymmetric stress states. It is proposed to present the solutions of the theory of elasticity, which are expressed by the product of the harmonic function of two variables to the degree of the coordinate. Polynomial solutions that depend on three coordinate variables are recorded. An example of the application of the proposed solution is given.*

Key words*: body, symmetric and asymmetric thermoelastic state, temperature field, stresses, displacement.*

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Introduction. Thermoelastic materials under the influence of various temperature fields are used in aerospace and other engineering [1–4]. Elastic bodies with applied power loads are widely used in power engineering, technological and engineering structures [5, 6].

Overview of known static solutions of the equations of the theory of elasticity and thermoelasticity and their application. The methods of solving static boundary value problems in the elastic three-dimensional body are based mainly on the construction and application of various representations of general solutions of the equations of the theory of elasticity [1, 2, 5, 6]. It is known [1, 6] that the representation of the solution of Lamé equations, which has been independently obtained by Papkovich and Neuber and contains four harmonic functions, is often used. In paper [7], the general solution of the equations of the theory of elasticity for the elastic isotropic body is constructed and it is proved that its stress state can be expressed in terms of three harmonic functions. The advantage of this solution is that the volume deformation is expressed only by one function. While investigating three-dimensional static problems of the theory of thermoelasticity [1–4, 6], known solutions of the equations of the theory of elasticity, with added specific temperature distribution of stresses given by the thermoelastic potential are used. The vast majority of partial solutions of Navier's equations of the theory of thermoelasticity are constructed using thermoelastic potential [1–4]. In paper [6], solutions of the equations of the theory of thermoelasticity for the linear distribution of temperature without thermoelastic potential application are presented.

In the paper [8], a three-dimensional stationary temperature field is considered and three-dimensional thermoelastic potential is used for the construction of a two-dimensional theory of thermoelastic plates. In the paper [9], temperature stresses in an elastic parallelepiped with free faces are investigated and a semi-analytical algorithm for the solution of the three-dimensional problem of thermoelasticity is proposed. In papers [10, 11] two-dimensional calculation models for the calculation of the stress state of elastic plates without plane stress hypotheses application are constructed on the basis of the general solution of the equations of the theory of elasticity [7].

The objective of the paper is to find physically substantiated partial temperature solutions of Navier's equations with partial derivatives and to construct general solutions of the theory of elasticity which describe the thermoelastic state of the body.

Statement of the problem and notation of static thermoelasticity equations. Let us consider the general formulation of the three-dimensional static problem of the theory of thermoelasticity for the three-dimensional isotropic body in the Cartesian coordinate system: $x_1 = x,\ x_2 = y,\ x_3 = z$. The initial temperature, when there are no stresses, is accepted equal to zero. The temperature in the body varies within such limits that the elastic and heat-conducting coefficients of the material can be modeled as constant.

Let's use the Duhamel–Neumann relation to represent thermoelastic stresses [3, 6] in homogeneous solid body

$$\sigma_k = 2G\left[\varepsilon_k + \frac{\nu}{1-2\nu}e - \frac{1+\nu}{1-2\nu}\alpha T\right],\ \tau_{kj} = G\gamma_{kj},\ k \neq j,\ k, j = \overline{1,3}, \tag{1}$$

where $G = E/2(1+\nu)$, E are shear and Young's modules, $\varepsilon_k = \frac{\partial \mathrm{u}_k}{\partial x_k}$ are elongation deformations, u_k are elastic displacements, $\gamma_{kj} = \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}$ are relative shear deformations, $e = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ is volume deformation, ν is Poisson's ratio, α is coefficient of thermal expansion.

Let us substitute relation (1) into the equilibrium equation of the thermoelastic body and write down the system of Navier's differential equations with partial derivatives for elastic displacements [1, 3, 6]

$$(1-2\nu)\nabla^2 u_k + \frac{\partial e}{\partial x_k} = 2(1+\nu)\alpha\frac{\partial T}{\partial x_k},\ k = \overline{1,3}, \tag{2}$$

where $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ is the Laplace operator. Let us assume that the body has the given stationary temperature field without internal heat release that satisfies the Laplace equation

$$\nabla^2 T = 0. \tag{3}$$

Functions that satisfy the Laplace equation (3) in three-dimensional or two-dimensional cases will be called harmonic functions.

Let us construct the solution of the equations system (2), and (3) by harmonic functions.

Presentation of partial solutions of thermoelasticity equations systems when the temperature is a three-dimensional or two-dimensional harmonic function

The system of Navier's equations of the theory of thermoelasticity (2) will be considered as the system of three-dimensional differential equations with non-zero right-hand parts, which are determined by the known three-dimensional harmonic temperature $T(x,y,z)$. The general solution of the equations system (2) is presented as the sum of homogeneous and partial solutions.

In order to find the partial temperature solution $\mathbf{u}^t$ of the equations system (2), let us analyze the known partial solutions of the equations of the theory of thermoelasticity. With index «t» at the top, we will denote the partial solutions of system (2), which depend on temperature. We assume that the surface of the body is not fixed, so free thermal elongation of the thermoelastic body can occur.

If the temperature is constant $T = T_0 = const$, then the partial solution of the equations system (1), (2) is

$$u_j^t = \alpha x_j T_0 , \; j = \overline{1,3}, \; e^t = 3\alpha T_0 . \tag{4}$$

When the temperature depends linearly only on one variable x: $T = t_1 x$, then the temperature displacements have the form [6]

$$u_1^t = \alpha(\frac{1}{2}xT - t_1 y^2), \; u_2^t = \alpha y T , \; u_3^t = \alpha z T , \; e^t = 3\alpha T . \tag{5}$$

From the above mentioned and physical considerations, the following patterns in the distribution of temperature displacements emerge: they are accumulate according to the integral law; if the temperature is directed to the constant value T_0, then the temperature displacements will coincide with the expressions (4); volume deformation of purely temperature displacements (4), (5) and known solutions obtained without the displacements thermoelastic potential of [2] are equal to: $e^t = 3\alpha T$. Let us construct partial solution of the differential system of equations (2)

$$u_j^t = \frac{\partial \vartheta}{\partial x_j} + \beta_1 \Omega_j , \; j = \overline{1,3}, \tag{6}$$

where T, $\Omega_j = \int \mathrm{T} dx_j$; $j = \overline{1,3}$ are harmonic functions, $\vartheta(x,y,z) = \beta(x\Omega_1 + y\Omega_2 + z\Omega_3)$ is biharmonic function, $\beta = -\frac{1}{6}\alpha$, $\beta_1 = \frac{4}{3}\alpha$. Indefinite integrals $\Omega_j = \int \mathrm{T} dx_j$ are defined in such a way as they are harmonic functions that are equal to zero when the temperature is zero. As we can see, temperature changes are expressed by integrals of temperature. If we direct the temperature to a constant value T_0, then the solution (6) coincides with expressions (4). According to the specified displacements (6), we determine the normal temperature deformations

$$e_j^t = \frac{\partial^2 \vartheta}{\partial x_j^2} + \frac{4}{3}\alpha T , \; j = 3, \; e^t = 3\alpha T . \tag{7}$$

Let's add the shear strains to relations (7) and substitute them together in relation (1) and find the temperature stresses

$$\sigma_j^t = 2G\{\frac{\partial^2 \vartheta}{\partial x_j^2} + \frac{1}{3}\alpha T\}, \; \Theta^t = 0,$$

$$\tau_{jm}^t = 2G\frac{\partial}{\partial x_j}\frac{\partial \vartheta}{\partial x_m} + \beta_1 G[\frac{\partial \Omega_j}{\partial x_m} + \frac{\partial \Omega_m}{\partial x_j}],\ m \neq j,\ j = \overline{1,3}. \quad (8)$$

Volume deformation and the sum of normal temperature stresses $\Theta^t = \sigma_1^t + \sigma_2^t + \sigma_3^t$ is determined by partial solution of the Navier's equation (6)–(8) and are equal to

$$e^t = \mathrm{div}\,\mathrm{u}_j^t = 3\alpha T,\ \Theta^t = \sigma_1^t + \sigma_2^t + \sigma_3^t = 0. \quad (9)$$

Displacements, deformations, and stresses are called temperature ones if they are determined by explicit partial solutions of the system of equations (2) and are equal to zero when the temperature is zero.

For example: partial solutions (4)–(6) are temperature solutions.

In order to simplify the presentation of the solution, it is reasonable to divide the stressed state of the thermoelastic body into symmetric (even) and asymmetric (odd) stress states relativele to variable z. According to the dependencies (1), (8), the even variable z temperature $T(x,y,-z) = T(x,y,z)$ generates an even variable z of normal σ_j^t, $j = \overline{1,3}$ and shear τ_{12}^t stresses, and correspondingly odd shear stresses τ_{j3}^t $j = \overline{1,2}$. For an odd temperature $T(x,y,-z) = -T(x,y,z)$, the normal stresses will be odd and the shear stresses τ_{j3}^t $j = \overline{1,2}$ will be even functions.

There are many problems where the temperature is modeled by the product of a harmonic function of two variables x, y on the degree of coordinate z. Let us consider these cases.

Construction of even partial solution for even temperature $T = T(x,y)$**.**

Let us take into account that the movement in the direction of axis Oz is $u_3^t = \alpha z T(x,y)$. Let's construct the even partial solution of the system of equations (2) when the temperature does not depend on variable z

$$u_j^t = \frac{\partial \vartheta}{\partial x_j} + \frac{3}{2}\alpha \Omega_j,\ j = \overline{1,2},\ u_3^t = \alpha z T,\ e^t = 3\alpha T, \quad (10)$$

where T, $\Omega_j = \int \mathrm{T} dx_j;\ j = \overline{1,2}$ are harmonic functions of variables x, y, $\vartheta(x,y) = -\frac{1}{4}\alpha(x\Omega_1 + y\Omega_2)$ is biharmonic function. If we direct temperature $T(x,y)$ to constant value T_0, then the solution (10) coincides with expressions (4). Temperature displacements u_j^t, $j = \overline{1,2}$ do not depend on the coordinate z.

According to the given displacements (10), we determine the deformations. Let's substitute these deformations into relation (1) and find the temperature stresses

$$\sigma_j^t = 2G\{\frac{\partial^2 \vartheta}{\partial x_j^2} + \frac{1}{2}\alpha T\},\ j = \overline{1,2},\ \sigma_3^t = 0,\ \Theta = 0,$$

$$\tau_{3j}^t = \alpha \mathrm{G} z\frac{\partial T}{\partial x_j},\ j = \overline{1,2},\ \tau_{12}^t = 2G\frac{\partial^2 \vartheta}{\partial x_1 \partial x_2}. \quad (11)$$

Therefore, in relations (11), only shear temperature stresses τ^t_{3j}, $j=\overline{1,2}$ depend linearly on the coordinate z. All other stresses do not depend on z and are described by two-dimensional harmonic functions.

Odd by variable z temperature $T = zT_1(x,y)$. In order to construct a partial solution of the system of equations (2), we will use representation (6), which after mathematical transformations will be reduced to the following form

$$u^t_j = \frac{\partial\Psi}{\partial x_j} + \beta_1\Omega_j\,,\ j=\overline{1,2}\,,\ u^t_3 = \beta[\frac{9}{2}x\int \mathrm{T}_1 dx + y\int \mathrm{T}_1 dy) - \frac{5}{2}\mathrm{z}^2\mathrm{T}_1]\,, \tag{12}$$

where expressions $\vartheta(x,y) = \beta z[(\frac{1}{2}x\int \mathrm{T}_1 dx + y\int \mathrm{T}_1 dy) + \frac{1}{2}\mathrm{z}^2\mathrm{T}_1]$, $\Omega_j = \mathrm{z}\int \mathrm{T}_1 dx_j$, $j=\overline{1,2}$, $e^t = 3\alpha T$ are modified, and all other coefficients are kept unchanged. According to the specified displacements (12), we determine the deformations. Let's substitute these deformations in relation (1) and find temperature stresses

$$\sigma_j = 2G[\frac{\partial^2\vartheta}{\partial x_j^2} + \frac{1}{3}\alpha T]\,,\ j=\overline{1,2}\,,\ \sigma^t_3 = -\frac{1}{3}G\alpha\mathrm{T}\,,\ \Theta = 0\,,$$

$$\tau^t_{j3} = G\beta\frac{\partial}{\partial x_j}[5x\int \mathrm{T}_1 dx + 2y\int \mathrm{T}_1 dy) - 2\mathrm{z}^2\mathrm{T}_1] + G\beta_1\int T_1 dx_j\,,\ j=\overline{1,2}\,, \tag{13}$$

$$\tau^t_{12} = 2G\frac{\partial^2\vartheta}{\partial x\partial y} + \beta_1 G[\frac{\partial\Omega_1}{\partial y} + \frac{\partial\Omega_2}{\partial x}]\,.$$

Therefore, for all obtained partial solutions, the following physically substantiated regularities are fulfilled:

$e^t = 3\alpha T$, $\Theta^t = 0$.

Presentation of the general solution of the equations of the theory of thermoelasticity

The homogeneous solution of the system of equations (2) coincides with the general solution of linear equations of the theory of elasticity. This solution is constructed in the paper [7] using three harmonic functions. The homogeneous solution of the system of equations (2) is determined by the following theorem:

Theorem [7]. The general solution of the system of equations of the theory of elasticity (2), when $T=0$, can be presented in the following form

$$\mathbf{u} = \mathrm{grad}P - 4(1-\nu)\Psi^1\mathbf{k} + \mathrm{r}ot(\Psi^2\mathbf{k})\,, \tag{14}$$

where $P = z\Psi^1 + \Psi^3$; $\Psi^j(x,y,z)$, $j=\overline{1,3}$ are three independent harmonic functions of three variables.

It should be noted that two functions Ψ^2, Ψ^3, do not influence the volume deformation value. Volume deformation is expressed only by function Ψ^1 according the formula

$$e=-2(1-2\nu)\frac{\partial\Psi^1}{\partial x_3}. \qquad (15)$$

Let us use the general representation of deformation (14). We add partial solutions (6) to it and obtain a general solution of the equations of the theory of thermoelasticity by four harmonic functions

$$u_j=\frac{\partial P}{\partial x_j}-(-1)^j\frac{\partial\Psi^2}{\partial x_{3-j}}+u_j^t,\ \ j=\overline{1,2},$$

$$u_3=\frac{\partial P}{\partial x_3}-4(1-\nu)\Psi^1+u_3^t. \qquad (16)$$

We add relation (8) to the general solution of the equations of the theory of elasticity in stresses [7, 11] and obtain a general expression of normal

$$\sigma_j=2G\left[\frac{\partial^2 P}{\partial x_j^2}-2\nu\frac{\partial\Psi^1}{\partial x_3}-(-1)^j\frac{\partial^2\Psi^2}{\partial x_1\partial x_2}\right]+\sigma_j^t,$$

$$\sigma_3=2G\left[\frac{\partial^2 P}{\partial x_3^2}-2(2-\nu)\frac{\partial\Psi^1}{\partial x_3}\right]+\sigma_3^t,\ \ j=\overline{1,2} \qquad (17)$$

and shearing

$$\tau_{j3}=G\left[\frac{\partial}{\partial x_j}\left[2\frac{\partial P}{\partial x_3}-4(1-\nu)\Psi^1\right]-(-1)^j\frac{\partial^2\Psi^2}{\partial x_{3-j}\partial x_3}\right]+\tau_{3j}^t,\ \ j=\overline{1,2},$$

$$\tau_{12}=G\left[2\frac{\partial^2 P}{\partial x_1\partial x_2}+\frac{\partial^2\Psi^2}{\partial x_2^2}-\frac{\partial^2\Psi^2}{\partial x_1^2}\right]+\tau_{12}^t \qquad (18)$$

stresses as the sum of elastic and temperature components [8]. Let us take into account formulas (9), (15), (17), and find the sum of normal stresses and volume deformation of thermoelastic body

$$\Theta=-2E\frac{\partial\Psi^1}{\partial z},\ \ e=-2(1-2\nu)\Theta+3\alpha T. \qquad (19)$$

The second equality (19) is also given in the paper [2].

In the case when the temperature depends on the product of a harmonic function of two variables on the degree of coordinate z, there is the need to use elastic solutions (14), which can also be presented in the form of products.

Let's specify the presentation of solution (14), when the displacements are described by the product of the variable z degree on the harmonic functions of two variables x, y. In this case, three-dimensional harmonic functions of three variables can be decomposed into series by variable z. If we restrict ourselves to the second degree of variable z, then we obtain

$$\Psi^k(x,y,z) = z^2\varphi_2^k + z\varphi_1^k + \varphi_0^k - x\int\varphi_2^k dx,\ k=\overline{1,3},\ P = z^3P_3 + z^2P_2 + zP_1 + P_0, \qquad (20)$$

where $\varphi_j^k(x,y)$, $j=\overline{0,2}$ are harmonic functions of two variables, $x\int\varphi_2^1 dx$ is a biharmonic function, $P_3 = \varphi_2^1$, $P_2 = \varphi_1^1 + \varphi_2^3$, $P_1 = \varphi_0^1 + \varphi_1^3 - x\int\varphi_2^1 dx$, $P_0 = \varphi_0^3 - x\int\varphi_2^3 dx$. Functions (20) are three-dimensional harmonic functions.

Harmonic polynomials $\Lambda_j^k(x, y, z)$ from three variables should be added to functions Ψ^k (20). Let's separate functions (20) and polynomials in such a way that they describe symmetric

$$\Psi^1 = z\varphi_1^1,\ \Psi^k(x,y,z) = z^2\varphi_2^k + \varphi_0^k - x\int\varphi_2^k dx,\ k=\overline{2,3},$$

$$P = z^2P_2 + P_0,\ \mathrm{e} = -2(1-2\nu)\varphi_1^1, \qquad (21)$$

$$\Lambda_2^k = a_{1m}^k x_i[\frac{1}{2}(z^2 + x_m^2) - \frac{1}{3}x_i^2] + a_{2m}^k x_i(z^2 - x_m^2),\ m \neq i \neq 3,$$

and asymmetric

$$\Psi^1 = z^2\varphi_2^1 + \varphi_0^1 - x\int\varphi_2^1 dx,\ \Psi^k(x,y,z) = z\varphi_1^k,\ k=\overline{2,3},$$

$$P = z^3P_3 + zP_1,\ \mathrm{e} = -4(1-2\nu)z\varphi_2^1, \qquad (22)$$

$$\Lambda_3^k = z[\frac{1}{2}(x^2+y^2) - \frac{1}{3}z^2]$$

stress states by variable z.

Let us substitute functions (21), (22) into expressions (16)–(18) and find the components of symmetric and asymmetric stress-strain states, respectively.

Let us give an example of practical problems solution due to the above mentioned formulas.

Torsion of elastic bodies. Let us consider the prismatic body

$$\coprod = \{(x,y,z) \in ([-a,a]\times[-b,b]\times[-h,h])\},$$

to the ends $z = \pm h$ of which only shear loads are applied to creating torques, and the side faces are free from loads. This is the well-known problem of rectilinear rod torsion [6], which is described by functions (22). For this problem, all normal stresses and temperatures are equal to zero. From this condition and relation (19), we define two-dimensional harmonic functions

$$\Psi^1 = 0,\ P = z\varphi_1^3,\ \Psi^2 = z\varphi_1^2. \qquad (23)$$

The known expression of elastic displacements for this problem is as follows:

$$u = -\theta zy,\ v = \theta zx,\ w = \theta\psi(x,y), \qquad (24)$$

where θ is the relative twist angle of the body cross-section, $\Delta\psi(x,y) = 0$.

In order to obtain ratio (24), displacement functions (22), (23) should be set in the following form

$$P = z\theta\psi(x,y)\,,\;\; \Psi^3 = z\varphi_1^3 = \theta z\psi\,,\;\; \Psi^2 = -z\int \frac{\partial}{\partial x}\varphi_1^3 dy - \theta\Lambda_3^2\,, \qquad (25)$$

and set all other functions equal to zero. Let us substitute the displacement functions (25) into relation (16) and obtain the representation (24). Let us substitute functions (25) into the expression of normal (17) and shear (18) stresses and make sure that normal stresses are equal to zero, and shear stresses are as follows

$$\tau_{j3} = G\theta[\frac{\partial}{\partial x_j}\psi + (-1)^j x_{3-j}]\,,\;\; j = \overline{1,2}\,,\;\; \tau_{12} = 0\,. \qquad (26)$$

Stresses (26) coincide with stresses found in the paper [6].

Discussion of results. We determined two important regularities of the temperature stresses distribution in the body: $e^t = 3\alpha T$, $\Theta^t = 0$. We will show that they are closely related. Let us substitute the expression $e^t = 3\alpha T$ in Duhamel-Neumann relation (1) and find a simple relationship between temperature stresses and strains

$$\sigma_k^t = 2G(\varepsilon_k^t - \alpha T)\,,\;\; k = \overline{1,3}\,. \qquad (27)$$

If we sum relation (27), we obtain $\Theta^t = 0$.

The thermoelastic potential of displacements Φ [1, 3] is widely used in the theory of thermoelasticity [1, 3]

$$\mathbf{u}^t = \mathrm{grad}\Phi\,, \qquad (28)$$

This potential is partial solution of the system of equations (2). Volume deformation of temperature displacements (28) is equal to

$$e^t = \Delta\Phi = \frac{1+\nu}{1-\nu}\alpha T\,. \qquad (29)$$

The value of volume deformation (29) does not comply with known solutions of thermoelasticity (4), (5) and expression (9). The thermoelastic potential of displacements Φ is defined implicitly as a partial solution of equation (29), which includes elastic displacements. The known partial solutions of thermoelasticity equations (4), (5) do not follow from the expression of thermoelastic displacements potential (28), (29).

Conclusions. It is determined that the sum of normal temperature stresses is equal to zero, and volume deformation is equal to $e = 3\alpha T$; if the temperature does not change in a certain spatial direction, then in this direction the normal temperature stresses are zero. A simple relationship (27) between temperature stresses and deformations is obtained. Symmetric and asymmetric by coordinate z elastic and temperature states of the body are constructed. Harmonic functions which describe displacements and stresses in these stressed-strained states are found. The obtained formulas are used and the solution to the well-known problem of prismatic elastic bodies torsion is recorded.

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ПОБУДОВА СТАТИЧНИХ РОЗВ’ЯЗКІВ РІВНЯНЬ ТЕОРІЇ ПРУЖНОСТІ Й ТЕРМОПРУЖНОСТІ

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Резюме. *Знайдено нові розв'язки теорій термопружності і пружності в декартовій системі координат. Для описування термопружного стану використано лінійну статичну модель тривимірного* *ізотропного тіла під дією стаціонарного температурного поля за відсутності об'ємних сил. Використано співвідношення Дюамеля–Неймана для подання термопружних напружень в однорідному твердому тілі. Після підстановки термопружних напружень в рівняння рівноваги термопружного тіла записано систему диференціальних рівнянь Нав'є в частинних похідних другого порядку на пружні переміщення. Загальний розв'язок системи рівнянь Нав'є наведено у вигляді суми однорідного йі часткового розв'язків. Побудовано й фізично обґрунтовано нові явні часткові розв'язки рівнянь термопружності, коли температурне поле задається тривимірними або двовимірними гармонічними функціями. Переміщення, деформації й напруження, які визначаються цими частковими розв'язками, названі температурними. Розглянута модель деформованого тіла базується на поданні переміщень і напружень через чотири гармонічні функції, три функції описують пружний стан і одна функція описує температурні деформації. Отримано просту формулу для вираження нормальних температурних напружень і показано, що їх сума дорівнює нулю. Також розглянуто окремі випадки термопружно-деформованого стану, коли температура залежить від добутку гармонічної функції від двох змінних на степінь координати z. Знайдено для них часткові й загальні розв'язки. Побудовано загальні розв'язки рівнянь термопружності (Navier's equations) через чотири гармонічних функції, коли температурне поле задається тривимірними або двовимірними гармонічними функціями. Термопружний стан тіла розділено на симетричний і несиметричний по координаті z напружені стани. Запропоновано подання розв'язків теорії пружності, які виражаються через добуток гармонічної функції від двох змінних на степінь координати z. Записано поліноміальні розв'язки, які залежать від трьох координатних змінних. Наведено приклад використання запропонованих розв'язків для визначення напружено-деформованого стану призми під дією зусиль скручування.*

Ключові слова: *тіло, симетричний і несиметричний термопружний стан,* температурне поле, *напруження, переміщення.*

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