МІНІСТЕРСТВО ОСВІТИ І НАУКИ УКРАЇНИ

ТЕРНОПІЛЬСЬКИЙ НАЦІОНАЛЬНИЙ ТЕХНІЧНИЙ УНІВЕРСИТЕТ
ІМЕНІ ІВАНА ПУЛЮЯ

МЕТОДИЧНІ ВКАЗІВКИ

до практичних занять і самостійної роботи студентів
з дисципліни
«ДИСКРЕТНА МАТЕМАТИКА»

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CONTENTS

INTRODUCTION

The course «Discrete mathematics» is in the list of natural and scientific subjects of formation of bachelors of specialties 123 «Computer Engineering», 122 «Computer Science and Informational Technologies» and is one of the basic subjects of mathematical courses of this cycle.

The proposed material consists of the following chapters: «Foundations of set theory»; «Combinatorics»; «Boolean algebra»; «Elements of graph theory».

To master the subject «Discrete mathematics», the student should have the knowledge of mathematics on the level of high school and some other main notions from the Higher Mathematics course.

The methodical instructions from the subject «Discrete mathematics» consist of references (manuals, tutorials and monographs), which can be used to more in-depth mastering of material, or for deeper studying of certain theoretical notions and practical examples.

1. FOUNDATIONS OF SET THEORY

1.1. Examples of tasks.

Task 1. Explain, why $3 \in \{1, 2, 3, 4\}$, but $\{1, 2\} \notin \{\{1, 2, 3\}, \{2, 3\}, 1, 2\}$.

Solution.

The set $\{1, 2, 3, 4\}$ consists of four elements, one of which is 3, therefore, one can write $3 \in \{1, 2, 3, 4\}$.

The set $\{\{1, 2, 3\}, \{2, 3\}, 1, 2\}$ consists of four elements: the set $\{1, 2, 3\}$, the set $\{2, 3\}$, object (the element of the set) 1 and object (the element of the set) 2. In such elements composition set $\{1, 2\}$ does not occur, therefore, $\{1, 2\} \notin \{\{1, 2, 3\}, \{2, 3\}, 1, 2\}$.

Task 2. Let the set $A = \{3, 6, 9, 12, 15, 18, 21, 24\}$ is given. Describe this set using the set-builder notation.

Solution.

The set A via the set-builder notation is defined as follows: $A = \{x \mid 0 < x \le 24 \text{ and } x \text{ is a multiple of } 3\}$.

Task 3. Prove, that the sets $A = \{2, 5, 4, 2\}$ and $B = \{5, 4, 2\}$ are equal.

Proof.

Two sets A and B are equal if and only if each element of A is an element of B and vice versa. For the given sets this condition is true, therefore, they are equal, namely, $A = B$.

Task 4. Prove that the empty set is a subset of any set.

Proof.

Let's assume that there exists the set A such that $\varnothing \not\subseteq A$. This means that $\varnothing$ has such element a that is not in A. And this is not possible since $\varnothing$ does not contain any element.

Task 5. Let $A = \{a, b, c, f, g, d\}$, $B = \{b, c, g, k, l\}$. Find $A \cup B$, $A \cap B$, $A \setminus B$, $A \Delta B$.

Solution.

$A \cup B = \{a, b, c, f, g, d, k, l\}$, $A \cap B = \{b, c, g\}$, $A \setminus B = \{a, f, d\}$, $A \Delta B = (A \cup B) \setminus (A \cap B) = \{a, f, d, k, l\}$.

Task 6. Let $M_1 = \{x \mid \sin x = 1\}$, $M_2 = \left\{x = \frac{\pi}{2} \pm 2\pi k\right\}$, where $k \in Z$. Prove that $M_1 = M_2$.

Solution.

Step 1. Let's show that $M_1 \subseteq M_2$.

$$\forall x \in M_1 \Rightarrow \sin x = 1 \Rightarrow x = \frac{\pi}{2} \pm 2\pi k,\ k \in Z \Rightarrow x \in M_2 \Rightarrow M_1 \subseteq M_2.$$

Step 2. Let's show that $M_2 \subseteq M_1$.

$$\forall x \in M_2 \Rightarrow x = \frac{\pi}{2} \pm 2\pi k, \quad k \in Z \Rightarrow \sin x = 1 \Rightarrow x \in M_1 \Rightarrow M_2 \subseteq M_1.$$

As a result of 1 and 2 one can conclude that $M_1 = M_2$.

Task 7. Prove that $A = (A \cap B) \cup (A \setminus B)$, where A and B are the sets.

Solution.

Step 1. Let's show that $A \subseteq (A \cap B) \cup (A \setminus B)$

$\forall x \in A \Rightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \notin B) \Rightarrow x \in A \cap B \text{ or } x \in A \setminus B \Rightarrow$

$\Rightarrow x \in (A \cap B) \cup (A \setminus B) \Rightarrow A \subseteq (A \cap B) \cup (A \setminus B).$

Step 2. Let's show that $(A \cap B) \cup (A \setminus B) \subseteq A$

$\forall x \in (A \cap B) \cup (A \setminus B) \Rightarrow x \in A \cap B \quad \text{or} \quad x \in A \setminus B \Rightarrow$

$\Rightarrow (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin B) \Rightarrow x \in A \text{ and } (x \in B \text{ or } x \notin B) \Rightarrow$

$\Rightarrow x \in A \text{ and } (x \in B \text{ or } x \notin B) \Rightarrow x \in A \Rightarrow (A \cap B) \cup (A \setminus B) \subseteq A$

As a result of 1 and 2 one can conclude that $A = (A \cap B) \cup (A \setminus B)$.

Task 8. Prove the following identity $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof.

Let $x \in A \cup (B \cap C)$, then $x \in A$ or $x \in B \cap C$. If $x \in A$, then x belongs to the union of A with any other set, namely, $x \in A \cup B$ and $x \in A \cup C$, therefore, x ϵ is an element of intersection $A \cup B$ and $A \cup C$, namely, $x \in (A \cup B) \cap (A \cup C)$.

If $x \in B \cap C$, then $x \in B$ and $x \in C$, therefore, $x \in A \cup B$ and $x \in A \cup C$, namely and in this case x is an element of intersection of this sets.

Thereby, it was proven that $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$. In the same way one can prove the statement $A \cup (B \cap C) \supset (A \cup B) \cap (A \cup C)$. According to the definition of identity of the sets we can get the necessary identity $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Task 9. Prove the identity $A \cup A = A$.

Proof.

The identity $A \cup A = A$ can be proven using the following transformations by employing the identities of the sets algebra:

$$A \cup A = (A \cup A) \cap U = (A \cup A) \cap (A \cup \overline{A}) = A \cup (A \cap \overline{A}) = A \cup \varnothing = A.$$

Task 10. List all subsets of the set $A = \{\{1, 3\}, \{2, 3, 5\}, 4\}$.

Solution.

The number of subsets can be determined via the formula 2^n, where n is the number of elements of the set A, therefore, $2^3 = 8$. Let’s list all the subsets of the set A:

$\varnothing, \{\{1,3\}\}, \{\{2,3,5\}\}, \{4\}, \{\{1,3\},\{2,3,5\}\}, \{\{1,3\},4\}, \{\{2,3,5\},4\}, \{\{1,3\},\{2,3,5\},4\}$.

Task 11. List the elements of the Cartesian product of two sets: $X = \{1, 2, 3\}$ i $Y = \{0,1\}$.

Solution.

$X \times Y = \{(1,0), (1,1), (2,0), (2,1), (3,1), (3,1)\}$;

$Y \times X = \{(0,1), (0,2), (0,3), (1,1), (1,2), (1,3)\}$.

Task 12. Let X is a set of the points of the segment $[0,1]$, and Y is the set of all points of the segment $[1, 2]$. Determine the set of points of $X \times Y$.

Solution.

$X \times Y$ is as set of points of the square $[0,1] \times [1, 2]$ with the vertices in the points $(0,1), (0, 2), (1,1), (1, 2)$.

Task 13. Find the domain and range of the following relations:

a) $\{(a,1), (a,2), (c,1), (c,2), (c,4), (d,5)\}$;

b) $\{(x, y) \mid x, y \in \mathrm{R} \text{ and } x = y^2\}$, where R is a set of real numbers.

Solution.

In the task a) the domain range of the relation is a set $\{a, c, d\}$, the range of the relation is a set $\{1, 2, 4, 5\}$; In task b) the domain range of the relation is a set $\{x \mid x \in R \text{ and } x \ge 0\}$, the range of the relation is a set R (the set of all real numbers).

Task 14. Let the following sets are given $X = \{x_1, x_2, x_3, x_4, x_5\}$; $Y = \{y_1, y_2, y_3, y_4\}$, as well as the following relation on this sets:

$R = \{ (x_1, y_1), (x_1, y_3), (x_2, y_1), (x_2, y_3), (x_2, y_4), (x_3, y_1), (x_3, y_2), (x_3, y_4), (x_4, y_3), (x_5, y_2), (x_5, y_4) \}$. Determine the quotient set $Y \setminus R$.

Solution.

It is obvious that $R(x_1) = \{y_1, y_3\}$; $R(x_2) = \{y_1, y_3, y_4\}$ and so on.

Let’s write down the quotient set according to all elements of the set X in the following form:

x_1	x_2	x_3	x_4	x_5
$\{y_1, y_3\}$	$\{y_1, y_3, y_4\}$	$\{y_1, y_2, y_4\}$	$\{y_3\}$	$\{y_2, y_4\}$

The union of all sets of the second row of the table form the quotient set $Y \setminus R = \{\{y_1, y_3\}, \{y_1, y_3, y_4\}, \{y_1, y_2, y_4\}, \{y_3\}, \{y_2, y_4\}\}$.

Task 15. Let the following set is given $A = \{\Pi, \Delta, O, \Omega\}$ i and let the relation $R \subseteq A \times A$ is defined in the following form:

$R = \{(\Pi,\Pi), (\Pi,\Delta), (\Pi,\Omega), (\Delta,\Pi), (\Omega,\Pi), (\Omega,\Omega), (O,\Omega), (O,O)\}$.

Determine whether R is an equivalence relation.

Solution.

R is not reflexive, namely, $\Delta \in A$, but $(\Delta,\Delta) \notin R$.

R is not symmetric, since $(O,\Omega) \in R$, але $(\Omega,O) \notin R$.

R is not antisymmetric, since $(\Delta,\Pi) \in R$ and $(\Pi,\Delta) \in R$, but $\Delta \neq \Pi$.

R is not transitive, since $(\Delta,\Pi) \in R$ i $(\Pi,\Omega) \in R$, but $(\Delta,\Omega) \notin R$.

Therefore, the relation R is not an equivalence relation.

Task 16. Let $A = \{-2, -1, 0, 1, 2\}$, and $B = \{0, 1, 2, 3, 4, 5\}$. The relation $f \subseteq A \times B$ is given as $f = \{(-2,5), (-1,2), (0,1), (1,2), (2,5)\}$. Determine whether this relation is a function.

Solution.

The relation f is a function from A to B, since $f \subseteq A \times B$, and each of its elements A is present as a first component of the tuple from f exactly once.

Task 17. Let A and B is a set of real numbers, and the function $f : A \to B$ is defined as $f(x) = x^2$. Check whether this function is surjective, injective and bijective?

Solution.

Function is not surjective, since there no such real number a, for which $f(a) = -1$. Function is not injective, since $f(2) = f(-2)$, but $2 \neq -2$. It can be noticed that if A and B are the sets of positive real numbers then the function $f : A \to B$ is injective and surjective.

1.2. Tasks for self-study

Task 1. Describe each of the following sets:

a) $A = \{x \in N \mid x \text{ is divisible by } 2 \text{ and } x \text{ is divisible by } 3\}$;

b) $A = \{x \mid x \in A \text{ and } x \in B\}$;

c) $A = \{x \mid x \in A \text{ and } x \notin B\}$.

Task 2. List the elements of the set $\{x \mid x \text{ is integer and } x^2 < 100\}$.

Task 3. List the elements of the set

$X = \{x \mid x \text{ is vowel of English alphabet}\}$.

Task 4. Describe the set $\{a, b, c, d, e, f, g, h, i, j\}$ using the set builder notation.

Task 5. List the subsets of the set $\varnothing$.

Task 6. List the subsets of the set $\{a, b, c, d\}$.

Task 7. Determine the number of elements in each set:

a) $\{\varnothing, \{\varnothing\}\}$; b) $\{\{\varnothing, \{\varnothing\}\}\}$; c) $\{\varnothing, \{\varnothing\}, a, b, \{a, b\}, \{a, b, \{a, b\}\}\}$;

d); $\{1, 2, 3, \{1, 2, 3\}\}$; e) $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$.

Task 8. Let the set of first 20 positive integers is the universal set. Write down the following subsets of this set: A is the subset of even numbers; B is the subset of odd numbers; C is the subset of squares of numbers; D is the subset of prime numbers.

Task 9. Determine whether the sets A and B are equal (if not, then why?):

а) $A = \{2, 5, 4\}$, $B = \{5, 4, 2\}$;

б) $A = \{1, 2, 4, 2\}$, $B = \{1, 2, 4\}$;

в) $A = \{2, 4, 5\}$, $B = \{2, 4, 3\}$;

г) $A = \{1, \{2, 5\}, 6\}$, $B = \{1, \{5, 2\}, 6\}$;

д) $A = \{1, \{2, 5\}, 6\}$, $B = \{1, 2, 5, 6\}$.

Task 10. Prove that $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$, where A and B are the sets.

Task 11. Let $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{4, 5, 6, 7, 8, 9, 10\}$, $C = \{2, 4, 6, 8, 10\}$, and $U = \{1, 2, 3, \ldots, 10\}$. Determine the sets: 1) $A \cup C$; 2) $A \cap B$; 3) $A \cup (B \cup C)$; 4) $(A \cap B) \cup C$; 5) $\overline{(A \cap B)}$; 6) $\overline{A} \cap \overline{B}$; 7) $A \Delta B$; 8) $A - B$.

Task 12. Determine whether there exist such sets A, B, C, that $A \cap B \neq \varnothing$, $A \cap C = \varnothing$, $(A \cap B) \setminus C = \varnothing$?

Task 13. Prove the following identity $(\overline{A} \cup B) \cap A = A \cap B$.

Task 14. Prove the identity $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Task 15. Prove the following identities using the equivalent transformations: $A \setminus (A \setminus B) = B \setminus (B \setminus A)$; $(A \setminus B) \setminus C = (A \setminus C) \setminus (B \setminus C)$. Check the result using the Venn's diagrams.

Task 16. What is the relation between A and B, if $A \setminus B = B \setminus A = \varnothing$?

Task 17. Show that the following identities are true:

a) $\overline{A \cap \overline{B}} \cup B = \overline{A} \cup B$; b) $(A \cap B \cap C) \cup (\overline{A} \cap B \cap C) = B \cap C$.

Task 18. Prove the following identities:

a) $A \cup (B \setminus A) = A \cup B$; b) $A \cap (B \setminus A) = \varnothing$; c) $A \setminus (A \cap B) = A \setminus B$;

d) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Task 19. Prove the equality of two sets using the Venn's diagrams $(\overline{A\cup B})=\overline{A}\cap\overline{B}$.

Task 20. Build the Venn's diagram for each of the given sets and mark theparts that corresponds to these sets:

1) $A-B$; 2)$(\overline{A\cap B})$; 3) $(A\cup B)-A\cap B$; 4) $A\cup(B\cap C)$; 5) $(B-C)-A$;

6) $B-(A\cup C)$; 7) $(\overline{A\cap B\cap C})$.

Task 21. Find the following sets a) $\varnothing\cap\{\varnothing\}$; b) $\{\varnothing\}\cap\{\varnothing\}$; c) $\{\varnothing,\{\varnothing\}\}\setminus\varnothing$; d) $\{\varnothing,\{\varnothing\}\}\setminus\{\{\varnothing\}\}$.

Task 22. Find the Cartesian product of the sets $X=\{\nabla,\infty,\sum\}$, $Y=\varnothing$ and $Z=\{a,b,c,d,e,f\}$.

Task 23. Let $B=\{x\in N\mid 1<x<4\}$ and $C=\{x\in N\mid x^2-4=0\}$, where N is the set of positive integers. Which elements comprise the sets $B\times C$ i $C\times B$?

Task 24.

Two sets are given $X=\{x_1,x_2,x_3,x_4,x_5,x_6\}$ i $Y=\{y_1,y_2,y_3,y_4\}$, and the binary relation is defined $A=\{(x_1,y_2),(x_2,y_1),(x_2,y_2),(x_4,y_2),(x_4,y_3),(x_5,y_1),(x_5,y_3)\}$.

For the given relation:

a) Determine the domain and the range;

b) Find the matrix and build a graph;

c) determine the symmetrical (inverse) relation A^{-1}.

Task 25. Let X is a set of students, Y is a set of subjects. The relation xAy, where $x\in X$ and $y\in Y$, means «the student x studies subject y». Describe the domain, range, and inverse relation.

Task 26. Given the relations R_1 and R_2 on the set $A=\{1,2,3,4\}$ ($R_1\subset A^2$ and $R_2\subset A^2$): $R_1=\{(1,1),(1,2),(1,3),(2,1)\}$; $R_2=\{(1,1),(1,3),(1,4),(2,2),(2,3)\}$.

Find the relation $R_1\cup R_2$, $R_1\cap R_2$, $R_1\setminus R_2$, $R_2\setminus R_1$, $\overline{R_1}$, $\overline{R_2}$, $R_1\times R_2$ and determine their cardinality.

Task 27. Let $A=\{1,2,3,4,5\}$, $B=\{6,7,8,9\}$, $C=\{10,11,12,13\}$, $D=\{\Omega,\Delta,\mathrm{O},*\}$ and $R\subseteq A\times B$, $S\subseteq B\times C$, $T\subseteq C\times D$ are defined as follows: $R=\{(1,7),(4,6),(5,6),(2,8)\}$, $S=\{(6,10),(6,11),(7,10),(8,13)\}$, $T=\{(11,\Delta),(10,\Delta),(13,*),(12,\Omega),(13,\mathrm{O})\}$. Determine $S\circ R$, $S\circ S^{-1}$, $(T\circ S)\circ R$.

Task 28. Let $A=\{a,b,c\}$, $B=\{1,2,3,4\}$, $P_1\subseteq A\times B$, $P_2\subseteq B^2$.

Set the relation $P_1=\{(b,2),(a,3),(b,1),(b,4),(c,1),(c,2),(c,4)\}$ and $P_2=\{(1,1),(1,2),(1,4),(2,2),(2,4),(3,3),(3,2),(3,4),(4,4)\}$ using the graphs, find the matrix of inverse relation $(P_1 \circ P_2)^{-1}$. Using the matrix of relation P_2, check whether it is reflexive, symmetric, antisymmetric, and transitive.

Task 29. Show that the given binary relation is reflexive, symmetric and not transitive: $P = \{(x, y) \mid x, y \in R,\ |x - y| \leq 1\}$

Task 30. For the given set $A = \{a, b, c, d, e\}$, S, T, U, V are the relation on A, where

$S = \{(a,a), (a,b), (b,c), (b,d), (c,e), (e,d), (c,a)\}$;

$T = \{(a,b), (b,a), (b,c), (b,d), (e,e), (d,e), (c,b)\}$;

$U = \{(a,b), (a,a), (b,c), (b,b), (e,e), (b,a), (c,b), (c,c), (d,d), (a,c), (c,a)\}$;

$V = \{(a,b), (b,c), (b,b), (e,e), (b,a), (c,b), (d,d), (a,c), (c,a)\}$

Which of the relations S, T, U, V is transitive? Which of the relation S, T, U, V is antisymmetric?

Task 31. Let the set $A = \{a, b, c, d, e\}$ is given. Describe the relation R, which is defined on the set A, that is reflexive and symmetric, and not transitive.

Task 32.

Let $A = \{1, 2, 3, 4, 5, 6\}$ and the relation $R \subseteq A \times A$ is the set $R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (1,2), (1,4), (2,1), (2,4), (3,5), (5,3), (4,1), (4,2)\}$. Show that the relation R is an equivalence relation.

Task 33. Which of the following relations R is the partial order on the set $A = \{a, b, c, d\}$?

a) A is the set of humans, and a relation R is defined as xRy, if x is older than y;

b) A is the set citizens of Ukraine, and a relation R is given as xRy, if x has the bigger ID number than y;

c) A – множина цілих чисел, R визначено як xRy, якщо $x \geq 2y$.

Task 34. Given the relation on the set of integers Z: $P = \{(x, y) \mid x, y \in Z;\ (x - y) < 1;\ 0 \leq x \leq 2;\ 0 \leq y \leq 2\}$.

Check whether this relation is a partial order.

Task 35. Let's consider the relation of perpendicularity of straight lines on the set of straight lines on the plane. Determine whether this relation is the equivalence relation on this set.

Task 36. Let's consider the relation of parallelism of straight lines on the set of straight lines on the plane. Determine whether this relation is an equivalence relation on this set.

Task 37. The following relations are given:

a) $y^2 = x^2 + 4$; b) $y^3 = x^3 + 4$; c) $y = 5$; г) $y = \sqrt{x^2 - 2}$.

Which of the following relations is the function, if x and y are real numbers, x belongs to the domain, and y belongs to the range?

Task 38.

Let A is a set of all citizens of Ukraine. Show, which of $f(x)$ are the functions is they mean: a) $f(x) = \text{father of } x$; b) $f(x) = \text{son of } x$; c) $f(x) = \text{brother of } x$; d) $f(x) = \text{husband of } x$.

Task 39. Let $f \subseteq \mathrm{R} \times \mathrm{R}$, where R is a set of real numbers. Find the domain and range of the functions:

a) $f(x) = \sqrt{x-3}$;

b) $f(x) = \dfrac{1}{\sqrt{x-2}}$;

c) $f(x) = \dfrac{1}{x^2+4}$;

d) $f(x) = \dfrac{1}{x^2-4}$;

e) $f(x) = |x|$.

Task 40. Let X is a set of the unordered triples (a, b, c) of natural numbers. The relation $f: X \to Y$ maps each such triple (a, b, c) into the sum $a+b+c$.

Find the preimage of each of the first six natural numbers.

Task 41. Find out which of the given functions, which domain and range coincide with the real number axis is injective, surjective, have the inverse function?

a) $f(x) = |x|$;

b) $f(x) = x^2 + 4$;

c) $f(x) = x^3 + 6$;

d) $f(x) = x + |x|$;

e) $f(x) = x(x-2)(x+2)$.

2. BOOLEAN FUNCTIONS AND THEIR TRANSFORMATIONS. NORMAL FORMS OF BOOLEAN FUNCTIONS REPRESENTATION

2.1. Examples of tasks

Task 1. Determine the cardinality of the set of binary words on which the Boolean function is defined $y = f(x_1, x_2, x_3, x_4, x_5, x_6)$.

Solution.

The number of arguments of the given Boolean function is equal to 6 ($n=6$). The cardinality of the set of binary words, on which the Boolean function is defines $y=f(x_1,x_2,x_3,x_4,x_5,x_6)$, is calculated by the formula $|B^n|=2^n$.

In total, the number of all binary words, on which the Boolean function $y=f(x_1,x_2,x_3,x_4,x_5,x_6)$ is defined, is equal to $|B^6|=2^6=64$ words.

Task 2. Determine the number of Boolean function which depends on the 5 Boolean variables.

Solution.

The number of all Boolean functions, which depends on n Boolean variables, is equal to 2^{2^n}, therefore, the number of all Boolean functions that depends on 5 Boolean variables $x_1,x_2,\ldots,x_5$, is equal to $2^{2^n}=2^{2^5}=2^{32}=4294967296$.

Task 3. Build the truth table of the Boolean function $f(x,y,z)=(x\sim y)\vee((y\to x)\downarrow\bar{z})$ and determine its ordinal number.

Solution.

Let's build the truth table of the Boolean function $f(x,y,z)=U=(x\sim y)\vee((y\to x)\downarrow\bar{z})$ (Table 2.1). Let's use the additional notations $A=(x\sim y)$ i $B=(y\to x)\downarrow\bar{z}$. Therefore, $U=A\vee B$.

Table 2.1 is a truth table of the Boolean function $f(x,y,z)=(x\sim y)\vee((y\to x)\downarrow\bar{z})$

x	y	z	$A=(x\sim y)$	$(y\to x)$	$\bar{z}$	$B=(y\to x)\downarrow\bar{z}$	$U=A\vee B$
0	0	0	1	1	1	0	1
0	0	1	1	1	0	0	1
0	1	0	0	0	1	0	0
0	1	1	0	0	0	1	1
1	0	0	0	1	1	0	0
1	0	1	0	1	0	0	0
1	1	0	1	1	1	0	1
1	1	1	1	1	0	0	1

The binary code, which corresponds to the values of this function, is equal to 11010011 (the last column of Table 2.1).

The binary number 11010011_2 has the following form in the decimal number system:

$$f(x,y,z)_{10}=2^7\cdot1+2^6\cdot1+2^5\cdot0+2^4\cdot1+2^3\cdot0+2^2\cdot0+2^1\cdot1+2^0\cdot1=128+64+0+$$
$$+16+0+0+2+1=192+19=211_{10}.$$

The ordinal number of the function is 211_{10} .

Task 4. Build the truth table for the binary function $f(x, y)$ with the ordinal number 14.

Solution.

Let’s find the binary number that corresponds to the decimal number 14.

Let’s write down this binary number as a sum of powers of number 2, namely, $14_{10} = 8+4+2 = 1\cdot 2^3 + 1\cdot 2^2 + 1\cdot 2^1 + 0\cdot 2^0 = 1110_2$.

Therefore, $f_{14}(x, y)$ corresponds to the binary number 1110_2.

Let’s build the truth table, for this let’s write down the obtained number in the column of function value in such a way, that the least significant digit was in the last row.

Table 2.2 – Truth table of the function $f_{14}(x, y)$

x	y	$f_{14}(x,y)$
0	0	1
0	1	1
1	0	1
1	1	0

Task 5. Check whether the formulas U and B are equivalent, if $B = x \sim z$, and $U = (x \sim y) \vee ((y \to x) \downarrow \bar{z})$?

Solution.

Let’s build the truth table for the formula U (Table 2.1) and the formula B (Table 2.3). Let’s check the equivalence of the formulas using these tables.

Table 2.3 – The generalized truth table of the functions, given by formulas U and B

x	y	z	B	U
0	0	0	1	1
0	0	1	0	1
0	1	0	1	0
0	1	1	0	1
1	0	0	0	0
1	0	1	1	0
1	1	0	0	1
1	1	1	1	1

The analysis showed that the truth tables of the functions does not coincide (the columns U and B differs), therefore, the formulas are not equivalent.

Task 6. Check whether the following identities are true:

a) $x \vee (y \sim z) = (x \vee y) \sim (x \vee z)$;

б) $x(y \sim z) = (xy) \sim (xz)$.

Solution.

a) Using the equivalent transformations, let's transform the right and left part of identity $x \vee (y \sim z) = x \vee yz \vee \bar{y}\bar{z}$. At first, let's transform the left side:

$$x \vee (y \sim z) = (x \vee y) \sim (x \vee z) = (x \vee y)(x \vee z) \vee \overline{(x \vee y)}\,\overline{(x \vee z)} =$$
$$= x \vee xz \vee xy \vee yz \vee \bar{x}\bar{y}\bar{x}\bar{z} = x \vee yz \vee \bar{x}\bar{y}\bar{z} = x \vee yz \vee \bar{y}\bar{z}.$$

The left and the right side of the identity are equal, therefore, the identity holds.

b) Let's transform the left side of identity:

$x(y \sim z) = x(yz \vee \bar{y}\bar{z}) = xyz \vee x\bar{y}\bar{z} = x(yz \vee \bar{y}\bar{z})$.

Let's transform the right side of identity:

$$(xy) \sim (xz) = xyxz \vee \overline{xy}\,\overline{xz} = xyz \vee (\bar{x} \vee \bar{y})(\bar{x} \vee \bar{z}) = xyz \vee \bar{x} \vee \bar{x}\,\bar{z} \vee \bar{x}\,\bar{y} \vee \bar{y}\,\bar{z} =$$
$$= xyz \vee \bar{x} \vee \bar{y}\bar{z} = \bar{x} \vee yz \vee \bar{y}\,\bar{z}.$$

The result of transformations is $x(yz \vee \bar{y}\bar{z}) \neq \bar{x} \vee (yz \vee \bar{y}\bar{z})$. This fact can be checked using the truth table (Table 2.4), denoting $A = x(yz \vee \bar{y}\,\bar{z})$, $C = \bar{x} \vee (yz \vee \bar{y}\,\bar{z})$.

Table 2.4 – The truth table of the functions A amd C

x	y	z	yz	$\bar{y}\bar{z}$	$yz \vee \bar{y}\bar{z}$	$A = x(yz \vee \bar{y}\,\bar{z})$	$C = \bar{x} \vee (yz \vee \bar{y}\,\bar{z})$
0	0	0	0	1	1	0	1
0	0	1	0	1	0	0	1
0	1	0	0	0	0	0	1
0	1	1	1	0	1	0	1
1	0	0	0	1	1	1	1
1	0	1	0	1	0	0	0
1	1	0	0	1	0	0	0
1	1	1	1	1	1	1	1

The columns A and C differs, therefore, the identity does not hold.

Task 7. Find the function, which is dual to the $f(x, y, z)$, if it is known that $f(x, y, z) = 1$ only in the interpretations (001), (011), (111).

Solution.

Let's build the truth table of the function $f(x,y,z)$ (Table 2.5). Let's generate the set of the inverse values (10101110) for the column of function $f(x,y,z)$ values. If one writes this set in the reverse order, one gets, therefore, the column of values of the dual function f^*.

Table 2.5 – Truth table of the dual functions

x	y	z	$f(x,y,z)$	$f^*(x,y,z)$
0	0	0	0	0
0	0	1	1	1
0	1	0	0	1
0	1	1	1	1
1	0	0	0	0
1	0	1	0	1
1	1	0	0	0
1	1	1	1	1

Task 8. Find the Boolean function, which is dual to the Boolean function $f = x \wedge (y \vee z \wedge (u \vee v))$.

Solution.

Using the rule of finding the dual formulas of Boolean algebra (the duality principle), namely, to change all conjunctions into disjunctions, all disjunctions into conjunctions, writing down the parentheses, where it is necessary, so that the order of operations remained the same as it was, one can find the dual Boolean function $f^* = x \wedge (y \vee z \wedge (u \vee v))^* = x \vee y \wedge (z \vee u \wedge v)$.

Task 9. The Boolean functions $f(x,y,z)$ and $g(x,y,z)$ are given by the truth tables (Table 2.6). Determine whether the given Boolean functions are self dual.

Table 2.6 – The truth table of the functions $f(x,y,z)$ and $g(x,y,z)$

x	y	z	$f(x,y,z)$	$g(x,y,z)$
0	0	0	0	0
0	0	1	1	1
0	1	0	1	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	0	0
1	1	1	1	0

Solution.

It can be seen from the Table 2.6 that each value of the Boolean function $f(x,y,z)$ is the negation of its symmetric value: for instance, the Boolean function is equal to zero on the interpretation $(0,0,0)$, namely, $f(0,0,0)=0$, the symmetric value is equal to one on the interpretation $(1,1,1)$, namely, $f(1,1,1)=1$.

Therefore, the function $f(x,y,z)$ is self dual.

For The Boolean function $g(x,y,z)$, there are some values of the function, that are not equal to the negation of their symmetric values, for instance, the Boolean function is equal to zero on the interpretation $(0,0,0)$, namely, $g(0,0,0)=0$, and the symmetric value of this function is also equal to zero on the interpretation $(1,1,1)$, namely, $g(1,1,1)=0$.

Therefore, the function $g(x,y,z)$ is not self dual.

Task 10. Write down the disjunctive decomposition of a function $f(x,y,z,t)=\overline{(x\wedge y\vee \bar{z})}\wedge t$ by the variables x, z.

Solution.

Let's use the theorem about the disjunctive decomposition of the Boolean function by k variables:

$$f(x,y,z,t)=\bigvee_{\sigma_1,\sigma_2} x^{\sigma_1}\wedge z^{\sigma_2}\wedge f(\sigma_1,y,\sigma_2,t)=x^0\wedge z^0\wedge f(0,y,0,t)\vee$$
$$\vee x^0\wedge z^1\wedge f(0,y,1,t)\vee x^1\wedge z^0\wedge f(1,y,0,t)\vee x^1\wedge z^1\wedge f(1,y,1,t)=$$
$$=\bar{x}\wedge\bar{z}\wedge f(0,y,0,t)\vee\bar{x}\wedge z\wedge f(0,y,1,t)\vee x\wedge\bar{z}\wedge f(1,y,0,t)\vee x\wedge z\wedge f(1,y,1,t)$$

Let's calculate:

$$f(0,y,0,t)=\overline{(0\wedge y\vee\bar{0})}\wedge t=0;$$
$$f(0,y,1,t)=\overline{(0\wedge y\vee\bar{1})}\wedge t=t;$$
$$f(1,y,0,t)=\overline{(1\wedge y\vee\bar{0})}\wedge t=0;$$
$$f(1,y,1,t)=\overline{(1\wedge y\vee\bar{1})}\wedge t=\bar{y}\wedge t.$$

Let's substitute $f(0,y,0,t)$, $f(0,y,1,t)$, $f(1,y,0,t)$, $f(1,y,1,t)$ in the formula for the disjunctive decomposition by the variables x, z:

$$f(x,y,z,t)=\bar{x}\wedge\bar{z}\wedge 0\vee\bar{x}\wedge z\wedge t\vee x\wedge\bar{z}\wedge 0\vee x\wedge z\wedge(\bar{y}\wedge t)=\bar{x}\wedge z\wedge t\vee x\wedge z\wedge\bar{y}\wedge t.$$

Task 11. Write down the conjunctive decomposition of a function $f(x, y, z) = xy \vee \bar{z}$ by the variable x.

Solution.

Let's use the theorem about the conjunctive decomposition of the Boolean function (by one variable):

$$f(x, y, z) = (x^{\bar{0}} \vee f(0, y, z)) \wedge (x^{\bar{1}} \vee f(1, y, z)) = (x \vee f(0, y, z)) \wedge (\bar{x} \vee f(1, y, z)) =$$
$$= (x \vee 0 \cdot y \vee \bar{z})(\bar{x} \vee 1 \cdot y \vee \bar{z}) = (x \vee \bar{z})(\bar{x} \vee y \vee \bar{z}).$$

Task 12. Represent the function $f(x, y, z) = (x \oplus y) \to yz$ in the form of the perfect disjunctive normal form and perfect conjunctive normal form.

Solution.

Let's build the truth table of the given function (Table 2.7).

Table 2.7 – The truth table of the function $f(x, y, z) = (x \oplus y) \to yz$

x	y	z	$(x \oplus y)$	$\bar{y}\bar{z}$	$yz \vee \bar{y}\bar{z}$
0	0	0	0	0	1
0	0	1	0	0	1
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	1	0	0
1	0	1	1	0	0
1	1	0	0	0	1
1	1	1	0	1	1

PDNF (f_{PDNF}) is built on the ones of the function:

$$f_{PDNF} = x^0 y^0 z^0 \vee x^0 y^0 z^1 \vee x^0 y^1 z^1 \vee x^1 y^1 z^0 \vee x^1 y^1 z^1 = \bar{x}\bar{y}\bar{z} \vee \bar{x}\bar{y}z \vee \bar{x}yz \vee xy\bar{z} \vee x\bar{y}\bar{z}.$$

PCNF (f_{PCNF}) is built on the zeroes of the function:

$$f_{PCNF} = (x^{\bar{0}} \vee y^{\bar{1}} \vee z^{\bar{0}})(x^{\bar{1}} \vee y^{\bar{0}} \vee z^{\bar{0}})(x^{\bar{1}} \vee y^{\bar{0}} \vee z^{\bar{1}}) = (x \vee \bar{y} \vee z)(\bar{x} \vee y \vee z)(\bar{x} \vee y \vee \bar{z}).$$

Task 13.

Write down the constituents of zero and one corresponding to the interpretations of the Boolean function of three variables.

Solution.

The constituent of zero and the constituent of one of Boolean function can be uniquely determined by the number of the corresponding interpretations. The constituent of zero of a function $f(x, y, z)$ is the elementary disjunction. The interpretation which turns this elementary conjunction into zero, turns into zero the function $f(x, y, z)$ as well. The constituent of one of the function $f(x, y, z)$ is the elementary conjunction. The interpretation, which turns into one this elementary conjunctions, turns into one the function $f(x, y, z)$ as well.

The constituents of zero and one of the function of three variables is given at Table 2.8.

Table 2.8 – The constituents of zero and one of the function of three variables $f(x, y, z)$

Number of interpretation	Interpretation			Constituent of one	Constituent of zero
	x	y	z		
0	0	0	0	$\bar{x}\bar{y}\bar{z}$	$x \vee y \vee z$
1	0	0	1	$\bar{x}\bar{y}z$	$x \vee y \vee \bar{z}$
2	0	1	0	$\bar{x}y\bar{z}$	$x \vee \bar{y} \vee z$
3	0	1	1	$\bar{x}yz$	$x \vee \bar{y} \vee \bar{z}$
4	1	0	0	$x\bar{y}\bar{z}$	$\bar{x} \vee y \vee z$
5	1	0	1	$x\bar{y}z$	$\bar{x} \vee y \vee \bar{z}$
6	1	1	0	$xy\bar{z}$	$\bar{x} \vee \bar{y} \vee z$
7	1	1	1	xyz	$\bar{x} \vee \bar{y} \vee \bar{z}$

Task 14.

Using the elementary transformations, transform the formula F into DNF $F = ((x \vee y\bar{z}t)((\bar{y} \vee t) \to x\bar{z}\bar{t}) \vee yz) \vee (\bar{x} \vee t)$.

Solution.

$$F = ((x \vee y\bar{z}t)((\bar{y} \vee t) \to x\bar{z}\bar{t}) \vee yz) \vee (\bar{x} \vee t) = ((x \vee y\bar{z}t)(\overline{(\bar{y} \vee t)} \vee x\bar{z}\bar{t}) \vee yz) \vee (\bar{x} \vee t) =$$
$$= ((x \vee y\bar{z}t)(y\bar{t} \vee x\bar{z}\bar{t}) \vee yz) \vee \bar{x} \vee t = xy\bar{t} \vee x\bar{z}\bar{t} \vee yz \vee \bar{x} \vee t = t \vee xy \vee x\bar{z} \vee yz \vee \bar{x} =$$
$$= t \vee \bar{x} \vee y \vee \bar{z} \vee yz = \bar{x} \vee y \vee \bar{z} \vee t.$$

Task 15. Build PDNF of the function $f(x,y,z) = xy \vee \overline{(x(\bar{y} \vee z) \vee yz)}$, using the rules of transformation of arbitrary formula of logic algebra into PDNF.

Solution.

Let’s use the rules of transformation of arbitrary formula of logic algebra into DNF.

Let’s move to the negation of variables by applying de Morgan law:

$$x\,y \vee \overline{(x\,(\bar{y} \vee z) \vee y\,z)} = x\,y \vee \overline{(x(\bar{y} \vee z))}\,(\overline{y\,z}) = xy \vee (\bar{x} \vee \overline{(\bar{y} \vee z)})\,(\bar{y} \vee \bar{z}) =$$
$$= xy \vee (\bar{x} \vee (y\bar{z}))(\bar{y} \vee \bar{z}).$$

Let’s build the disjunctive normal form, using the distributive law, the laws of idempotency and of contradiction:

$$x\,y \vee (\bar{x} \vee (y\,\bar{z}))(\bar{y} \vee \bar{z}) = x\,y \vee (\bar{x}\,\bar{y} \vee y\,\bar{z}\,\bar{y} \vee \bar{x}\,\bar{z} \vee \bar{y}\,\bar{z}\,\bar{z}) = x\,y \vee \bar{x}\,\bar{y} \vee 0 \vee \bar{x}\,\bar{z} \vee y\,\bar{z} =$$
$$= xy \vee \bar{x}\bar{y} \vee \bar{x}\bar{z} \vee y\bar{z}.$$

The Boolean function depends on three variables; therefore, the missing variables should be introduced into the elementary, by applying the law of the excluded third:

$$x\,y \vee \bar{x}\,\bar{y} \vee \bar{x}\,\bar{z} \vee y\,\bar{z} = x\,y(z \vee \bar{z}) \vee \bar{x}\,\bar{y}(z \vee \bar{z}) \vee \bar{x}(y \vee \bar{y})\,\bar{z} \vee (x \vee \bar{x})\,y\,\bar{z}.$$

Using the distributive law, let’s open the parentheses and reduce the equal terms in order to get PDNF:

$$xyz \vee xy\bar{z} \vee \bar{x}\bar{y}z \vee \bar{x}\bar{y}\bar{z} \vee \bar{x}y\bar{z} \vee \bar{x}\bar{y}\bar{z} \vee xy\bar{z} \vee \bar{x}y\bar{z} = xyz \vee xy\bar{z} \vee \bar{x}\bar{y}z \vee \bar{x}\bar{y}\bar{z} \vee \bar{x}y\bar{z}.$$

The obtained perfect disjunctive normal form of the given Boolean function:

$$f(x,y,z) = xyz \vee xy\bar{z} \vee \bar{x}\bar{y}z \vee \bar{x}\bar{y}\bar{z} \vee \bar{x}y\bar{z}.$$

Task 16. Build the algorithm of a transition from the truth table of Boolean function into the PCNF.

Solution.

For the transition from the truth table of the Boolean function to the DCNF one can use the following algorithms:

a) find all the interpretations in the truth table of Boolean function, on which the values of function are equal to zero;

b) write down all the constituents of zero, that correspond to the obtained interpretations;

в) obtain all PCNF of a function using the join by conjunction operation of the written constituents of zero.

2.2. Tasks for self-study

Task 1. In how many times greater there are different binary words should be analyzed for the Boolean function $y = f(x_1, x_2, x_3, x_4, x_5)$, than for the Boolean function $y = f(x_1, x_2, x_3)$?

Task 2. In how much many times greater one can built the Boolean function, that depends on 6th variables, than from the 4th variables?

Task 3. Build the truth table of the following Boolean function and determine their ordinal number:

a) $f(x, y) = (x \sim y) \wedge (y \to x)$;

b) $f(x, y, z) = (x \wedge y) \oplus (x \wedge z) \oplus (y \wedge z)$;

c) $f(x, y, z) = (x \wedge y) \vee (x \wedge \bar{z}) \vee (y \wedge z)$;

d) $f(x, y) = (x \downarrow y) \wedge (y \to x)$;

e) $f(x, y, z) = (x \wedge y) \vee (x \mid \bar{z}) \vee (y \sim z)$.

Task 4. Check the following identities using the truth tables:

a) $x \vee (y \sim z) = (x \vee y) \sim (x \vee z)$;

b) $x \to (y \sim z) = (x \to y) \sim (x \to z)$;

c) $x \to (y \vee z) = (x \to y) \vee (x \to z)$;

d) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$;

e) $x \to (y \to z) = (x \to y) \to (x \to z)$.

Task 5. Setting $x_1 = 1$; $x_2 = 0$; $x_3 = 0$; $x_4 = 0$, find the values of the functions $(x_1 \vee x_2) \sim x_2 \bar{x}_3$ and $x_1 \bar{x}_2 \to (x_2 \sim x_3)$.

Task 6. Prove that the implication and equivalence can be determined via the other functions: $x_1 \to x_2 = \overline{x_1} \vee x_2$; $x_1 \sim x_2 = (x_1 \vee \overline{x_2})(\overline{x_1} \vee x_2)$.

Task 7. Using the main equivalences, prove the equivalence of the formulas U and B, if $U = (x \to y) \to ((x\bar{y}) \oplus (x \sim \bar{y}))$, $B = (x \vee y)(\bar{x} \vee \bar{y})$.

Task 8. Find the dual formulas for the following functions:

a) $(x \wedge (y \vee z)) \vee \bar{x} \wedge \bar{y}$;

b) $x \wedge y \vee y \wedge z \vee x \wedge z$;

c) $x\bar{y} \vee x \vee y \vee zt$.

Task 9. Determine whether the following functions are self dual:

a) $f(x, y) = (\bar{x} \vee y) \wedge (\bar{y} \vee x)$;

b) $f(x, y, z) = (\bar{x} \wedge \bar{y}) \vee (x \wedge z) \vee (\bar{y} \wedge \bar{z})$;

c) $f(x, y) = x \vee \overline{(x \wedge y)}$.

Task 10. Reduce the following expressions using the laws of Boolean logic. Then compare the following expressions with the given using the truth tables:

a) $(x \vee (\bar{t} \wedge y)) \wedge ((\bar{x} \wedge (\bar{y} \vee t)) \vee z)) \vee \bar{z} \vee (x \vee (y \wedge \bar{t}))$;

b) $((\bar{y} \vee \bar{z}) \wedge (x \vee y)) \vee (t \wedge \bar{z}) \vee (((\bar{y} \wedge \bar{x}) \vee z) \wedge (x \vee y))$;

c) $((x \vee z) \wedge (x \vee t)) \wedge (((z \vee (z \wedge y)) \wedge \bar{z}) \vee \bar{x})$.

Task 11. The Boolean function $f(x, y, z)$ is determined in the following way: it is equal to 1 when $x = 1$, or, if y and z take different values, and the value of x is less than the value of the variable z. In all other cases, the function is equal to 0. Build the truth table of the function $f(x, y, z)$ and write down the set $Q = \{(xyz) \mid (xyz) \in \mathrm{R}^3 \text{ and } f(x, y, z) = 1\}$.

Task 12. Find the disjunctive decomposition of the following Boolean functions by the variables x, z:

a) $(\bar{x} \vee (z \vee yz))(z \vee x\bar{z} \vee \bar{y})$;

b) $((x \vee y)(\bar{y} \vee \bar{z}) \vee (\bar{z} \vee x)) \vee (\bar{x} \vee z)$;

c) $(x \vee \bar{z})(\bar{x}t \vee yt \vee \bar{x}\bar{t} \vee y\bar{t})(x \vee z)$.

Task 13. Find the conjunctive decomposition of the following Boolean functions by the variables x, z:

a) $(x \vee \bar{z} \vee y)(x\bar{y} \vee \bar{x}\bar{y} \vee z)(x \vee \bar{y})$;

b) $((y \vee z)(t \vee y\bar{z})) \vee \bar{t}\bar{x} \vee ((z \vee y)(\bar{t} \vee \bar{z}))$;

c) $yt \vee ((z \vee \bar{t})(\bar{x} \vee z)(\bar{t} \vee \bar{z})(x \vee \bar{z})) \vee \bar{y}t$.

Task 14. Find the disjunctive decomposition of the Boolean function $f(x, y, z, t) = \overline{(x \wedge y \vee \bar{z})} \wedge (t \vee x)$ by the variable x.

Task 15. Find the constituents of zero and ones of the Boolean function that correspond to the interpretation of the function of four variables.

Task 16. Using the equivalent transformations, transform the following formulas into DNF: a) $F = (x \vee y\bar{z})(x \vee z)$; b) $F = (x \to y) \oplus (xy \vee z)$.

Task 17. Represent the following functions in the form of PDNF and PCNF:

a) $\sigma_f = (10001110)$, where σ_f is the column of the values of function from the truth table;

b) $\sigma_f = (01100110)$; c) $f(x, y, z) = (x \to y) \oplus z$; d) $f(x, y) = (x \downarrow y)(x \to y)$;

e) $f(x, y, z, t) = ((x \to y) \oplus z) \wedge \bar{t}$; e) $f_{201}(x, y, z)$.

Task 18. Construct the algorithm of transition from the truth table to the PDNF of the given function.

Task 19. Using the transformations in the form $A = Ax \vee A\bar{x}, \quad A \vee A = A$ move from the given DNF $f(x_1, x_2, x_3) = x_1\overline{x_2} \vee \overline{x_1}x_3$ to the PDNF.

Task 20. Write down the PDNF for the function $f(x_1, x_2, x_3)$, that has the null values on all odd binary interpretations.

Task 22. Let the function $f(x, y, z)$ is given in the following way: $f(x, y, z) = 0$, when $y = 1$ or $y < z$, and $f(x, y, z) = 1$ elsewhere. Using the truth table of the function write down the sey Q_f such that $Q_f = \{\sigma \mid \sigma \in \mathrm{R}^3 \text{ and } f(\sigma) = 0\}$ and write down the DCNF and DDNF of the given function.

3. GRAPH THEORY

3.1. Examples of tasks

Task 1. Several persons (more than two) take part in the chess contest. In some moment it was revealed that only two chess players played the same number of games. Prove that in this case there is either only one player, that have played no game or only one which have played all games.

Solution.

Using the language of graph theory the task can be formulated as follows. The graph with *n* (*n* > 2) vertices has only two vertices of the same degree. Prove that there is either one vertex of degree 0 or only one vertex of degree n-1. Let's consider all the possible negations of this statement. If one assumes that there are no vertices of the degree 0, as well as of degree *n*–1, then *n* vertices have the degree from 1 to *n*–2, therefore, there are either two pairs of vertices or three vertices of the same degree, that contradicts the statement. Therefore, there exists the vertices of either the degree zero or the degree *n*–1. Simultaneously this cannot be true. If there are two vertices of degree 0, then there remain *n*–2 vertices of pairwise different degrees from 1 to *n*–3, that is not possible. Also, there is not possible that when two vertices have the degree *n*–1 the other *n*–2 vertices have the pairwise different degrees from 2 to *n*–2.

Task 2. 29 teams take part in a football championship in one turn. Prove that in any moment of time there exists the team which played the even number of games.

Solution.

In terms of the graph theory the task is as follows: prove that in any graph with 29 there can be found one vertex of even degree. Considering that number 29 is odd, and the number of vertices of odd degree is even then one can found at least one vertex of even degree.

Task 3. Does the complete graph exists with the number of edges equal to

(a) 15; (b) 18; (c) 199…900…0 (k nines and k zeroes)?

Solution.

a) The number of edges of the complete graph with n vertices is equal to $n(n-1)/2$. The equation $n(n-1)/2 = 15$ has the following roots 6 and –5. Therefore, there exists the complete graph with 6 vertices, the number of edges of which is equal to 15.

b) The equation $n(n-1)/2 = 18$ does not have natural roots, therefore, such complete graph does not exist.

c) Let's consider the equation $n(n-1) = 2\cdot199\ldots900\ldots0 = 200\ldots0\cdot199\ldots9$ (k nines and k zeroes). The number in the right hand side of this equation can be expressed in the form $2\cdot10^k(2\cdot10^k - 1)$, therefore, $n = 2\cdot10^k$. Thus, such complete graph exists and has $2\cdot10^k$ vertices.

Task 4. Determine whether there exists the graph with six vertices, which degrees are equal to:

a) 1, 1, 2, 3, 4, 4 b) 0, 0, 2, 3, 3, 4.

Solution.

a) Such graph does not exist, since the sum of degrees of all its vertices is odd.

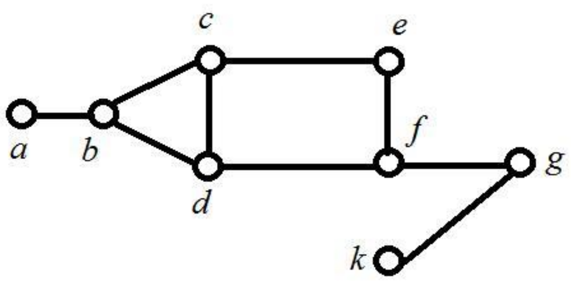
б) The graph with such degrees of vertices does not exist. According to the statement it has two isolated vertices (of degree zero), therefore, the maximum value of the degree of any of the other four vertices is 3, thus, this graph cannot have the vertex of degree 4.

Task 5. How to determine the degree of certain vertex of graph G using its adjacency matrix A?

Solution.

The degree of vertex v with the number i is equal to the sum of elements of ith row of matrix A, since this sum determines the number of vertices, adjacent with the vertex v.

Task 6. For the given graph construct the distance matrix. Find its center, radius, and diameter of the graph.



Solution.

The distance matrix has the following form:

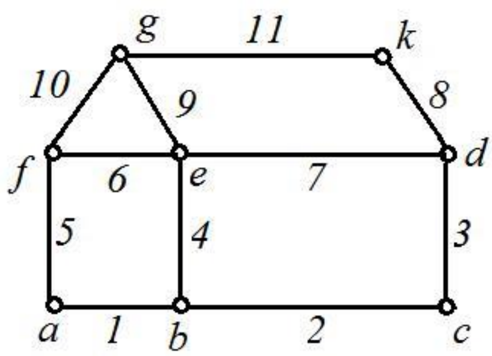
	a	*b*	*c*	*d*	*e*	*f*	*g*	*k*
a	0	1	2	2	3	3	4	5
b	1	0	1	1	2	2	3	4
c	2	1	0	1	1	1	2	3
d	2	1	1	0	2	1	2	3
e	3	2	1	2	0	1	2	3
f	3	2	1	1	1	0	1	2
g	4	3	2	2	2	1	0	1
k	5	4	3	3	3	2	1	0

In order to determine the center of the graph, let's find the maximum distance from each of the vertices of graph. Let's use the obtained distance matrix:

$$l(a)=5,\ l(b)=4,\ l(c)=3,\ l(d)=3,\ l(e)=3,\ l(f)=3,\ l(g)=4,\ l(k)=5.$$

The lowest maximal distance from the vertices *c*, *d*, *e*, *f*, therefore, the center of the graph is a set of vertices {*c*, *d*, *e*, *f*}. $R(G) = 3$, $D(G) = 5$.

Task 7. Graph is given



Determine the degrees of graphs vertices. Represent a graph using incidence matrix, adjacency matrix, and list of edges.

Solution.

Let's determine the degrees of all graph vertices:

$deg(a) = 2$; $deg(b) = 3$; $deg(c) = 2$; $deg(d) = 3$; $deg(e) = 4$; $deg(f) = 3$; $deg(g) = 3$; $deg(k) = 2$.

$$\sum deg\ v_i = 2+3+2+3+4+3+3+2 = 22 = 2\cdot 11$$

The incidence matrix has the following form:

	1	2	3	4	5	6	7	8	9	10	11
a	1	0	0	0	1	0	0	0	0	0	0
b	1	1	0	1	0	0	0	0	0	0	0
c	0	1	1	0	0	0	0	0	0	0	0
d	0	0	1	0	0	0	1	1	0	0	0
e	0	0	0	1	0	1	1	0	1	0	0
f	0	0	0	0	1	1	0	0	0	1	0
g	0	0	0	0	0	0	0	0	1	1	1
k	0	0	0	0	0	0	0	1	0	0	1

The adjacency matrix has the form:

	a	*b*	*c*	*d*	*e*	*f*	*g*	*k*
a	0	1	0	0	0	1	0	0
b	1	0	1	0	1	0	0	0
c	0	1	0	1	0	0	0	0
d	0	0	1	0	1	0	0	1
e	0	1	0	1	0	1	1	0
f	1	0	0	0	1	0	1	0
g	0	0	0	0	1	1	0	0
k	0	0	0	1	0	0	1	0

The list of edges has the form:

Edge		1	2	3	4	5	6	7	8	9	10	11
Vertex	s	*a*	*b*	*c*	*b*	*a*	*f*	*d*	*d*	*e*	*f*	*k*
	e	*b*	*c*	*d*	*e*	*f*	*e*	*e*	*k*	*g*	*g*	*g*

3.2. Tasks for self-study

Task 1. Suppose one has the set *V* that contains 3 documents. Build the graph of the relation $a \in A$ that characterizes the presence of document *a* in the set *A*, where *A* takes all subsets of the set *V*.

Task 2. Suppose the graph $G = (V, E)$ is given:

a) $V = \{1,2,3,4\}$, $E = \{(1,3),(2,3),(3,4),(4,1),(4,2)\}$;

b) $V = \{a,b,c,d,e\}$,$E = \{(a,d),(b,c),(b,e),(c,e),(d,b),(d,e),(e,a)\}$;

c) $V = \{1,2,3\}$, $E = \{(1,2),(1,3),(2,3)\}$;

Build the diagram, adjacency and incidence matrices for each of the given graph.

Task 3. Prove that in any graph $G = (V, E)$ $\sum_{v \in V} \delta(v) = 2|E|$.

Task 4. How many edges are in the graph with n vertices is all its vertices have the degree 2?

Task 5. Prove that in any graph the number of vertices of odd degree is even.

Task 6. Suppose there are p vertices of degree t in graph G with n vertices and m edges, and all other vertices has the degree $t+1$. Prove that $p = (t+1)\,n - 2m$.

Task 7. In certain company of n persons each person is acquainted with k and only k other persons. Whether some company is possible for

a) $n = 5$, $k = 2$; b) $n = 5$, $k = 3$.

Task 8. Prove that there exists at least two vertices of the same degree in any graph with n vertices ($n \geq 2$).

Task 9. Build the graph with five vertices, which has only two vertices of the same degree.

Task 10. Graph with five vertices has two vertices of the same degree. Can these two vertices have the degree 0 or the degree 4?

Task 11. Given the subset of the domino tiles without duplicates. Formulate the conditions when one can assemble unique continuous chain from this tiles using the rules of this game. How the answer differs when duplicates are allowed?

Task 12. Prove that there always exist three vertices which are either mutually adjacent or pairwise not adjacent in the arbitrary graph G with six vertices. (This is the mathematical formulation of the known task: prove that in a group of any six persons there exists either three persons which are not pairwise acquainted or three persons which are pairwise not acquainted).

Task 13. Determine whether exists the graph with six vertices, which degrees are equal to

a) 2,3,3,4,4,4; b) 2,2,2,4,5,5?

Substantiate the answer.

Task 14. Prove that all isomorphic graphs have the same number of vertices and the same number of edges.

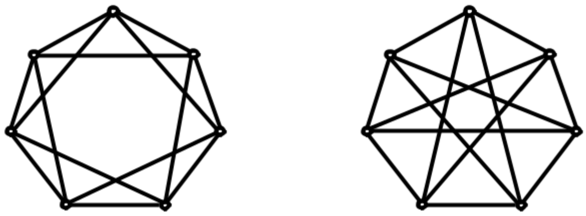
Task 15. Prove that in the isomorphic graphs the number of vertices of degree k is the same for the arbitrary k ($k \geq 0$).

Task 16. Prove that in the case when for some k ($k \geq 0$) the number of vertices of degree k in the graphs G_1 and G_2 differs, then the graphs G_1 are G_2 not isomorphic.

Task 17. Prove that in the isomorphic graphs the number of simple cycles of length l is the same for arbitrary l.

Task 18. Prove that the pairs of graphs that are shown on figure are isomorphic.

a) 

b) 

Task 19. Check whether the graphs G_1 and G_2 are isomorphic, that are given by their adjacency matrices A_1 and A_2.

$$A_1 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Task 20. Prove that the graphs G_1 and G_2 are isomorphic if and only if the adjacency matrix (incidence matrix) of one of the graphs can be obtained from the adjacency matrix (incidence matrix) of the other graph by means of corresponding permutations of rows and columns.

Task 21. The graph G is shown on the Figure.

a) Find all the trails which goes from the vertex v_1 to vertex v_{12}.

b) Find all the simple trails which goes from vertex v_1 to vertex v_{12}.

c) Find the trail that goes from the vertex v_1 to vertex v_{12} and contains all vertices of graph G.

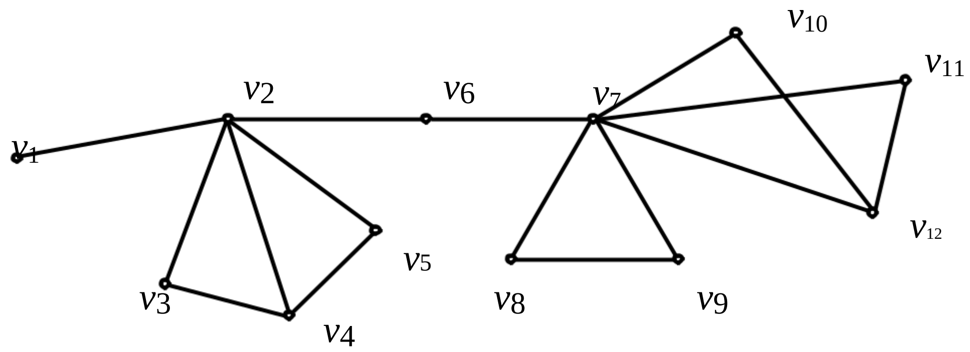
d) Determine whether there exists the simple trail in graph G that goes from the vertex v_1 to vertex v_{12} and contains all the vertices of graph G?

e) Find the walk in the graph G, that leads from vertex v_1 to vertex v_{12} and is not a trail.

f) Find the walk in the graph G, which goes from vertex v_1 to vertex v_{12}, contains all vertices of graph G and is not a trail.

g) Find any cycle in the graph G.

h) Find all simple cycles of the graph G.



Task 22. For the graph G, shown on the Figure, determine:

a) the distances $d\,(v_1,v_{12})$, $d\,(v_2,v_8)$, $d\,(v_{10},v_{11})$;

b) the eccentricity of all vertices;

c) the diameter $D(G)$;

d) the radius $R(G)$;

e) the central vertices;

f) the center.

Task 23. Find the cycles of the given length in graph K_5

a) 3; b) 4; c) 5.

Which of these cycles are simple?

Task 24. Determine whether exists the cycle of length 9 in the graph K_5? Justify your answer.

Task 25. Prove that any trail which leads from vertex v to the vertex w ($v \neq w$) contains the simple trail that leads from v to w.

Task 26. Disprove the statement: is some trail that goes from vertex v to vertex w, contains the vertex u ($u \neq v$ i $u \neq w$), than this trail contains the simple trail that goes from v to w and passes via u.

Task 27. Prove that if for arbitrary three vertices v, w and u in the graph G there exists the trails, the first one goes from v to w, and the second goes from w to u, then there is a trail in graph G that goes from v to u.

Task 28. Disprove the statement: if there exist the simple trails in the graph G for some three various vertices v, w and u, one of which goes from v to w, and the second from w to u, then there exist the simple trail in the graph G that goes from v to u and passes through w.

Task 29. Prove that any shortest trail that leads from the vertex v to the vertex w ($v \neq w$) is a simple trail.

Task 30. Prove that arbitrary cycle contains the simple cycle.

Task 31. Prove that when there are only two vertices of the odd degree in the graph G then these vertices are connected in the graph G.

Task 32. What the diameter and radius

a) of the complete graph K_n; b) of the complete bipartite graph $K_{n,m}$?

Task 33. Determine the chromatic number of

a) the complete graph K_n;

b) of the complete bipartite graph $K_{n,m}$;

c) of an arbitrary bipartite graph;

d) of the simple cycle of the length $2k$;

e) of the simple cycle of the length $2k+1$, $k \in N$.

Task 34. Determine the examples of graph which is Euler graph, but not Hamiltonian, and also of the Hamiltonian graph, which is not Euler.

Task 35. Prove that for the arbitrary oriented graph $G = (V,E)$ the next statements are true:

a) $\sum\limits_{v \in V} \delta^{+}(v) = |E|$; b) $\sum\limits_{v \in V} \delta^{+}(v) = \sum\limits_{v \in V} \delta^{-}(v)$.

Task 36. Does the oriented graph with three vertices exist, in which the out-degree are equal to 2, 2, and 0, and the respective in-degree are equal to –2, 1and 1?

Task 37. Determine the minimum and maximum possible number of edges in the connected bipartite graph with n vertices.

Task 38. Let $G = (V,E)$ is the connected graph which is not a complete graph. Prove that there are 3 vertices u, v and w in the graph G such that $(u,v) \in E$ and $(v,w) \in E$, though $(u,w) \notin E$.

Task 39. Prove that the diameter of graph G is equal to 1 if and only if the graph G is complete.

Task 40. Justify the following way of finding the central vertices of the given tree T. Choose an arbitrary terminal vertex and find the one of the most distant from it. It is obvious that the vertex v is also terminal. Then let's find the vertex w that is most distant from v; let's build a simple trail L from v to w. This is the longest simple trail in tree T. If the length of L is even, then the middle vertex of L is the only central vertex of the tree T. If the length L is odd then both vertices of the middle edge in L are central for the tree T.

4. COMBINATORICS

4.1. Examples of tasks

Task 1. Given the set {1, 2, 3, 4, 5, 6, 7}. Build 10 arrangements of 4 elements from this set that are lexicographically next after the 4517.

Solution.

Let's find in the arrangement 4517 the first number from the right which can be increased using the numbers which are to the right from it or using the numbers {2, 3, 6} which are not in this arrangement.

The number 7 does not suit, since it is the largest. Therefore, one increases the number «1» which can be substituted into one of the numbers {2, 3, 6, 7}. On the place of one we can put the smallest of these numbers, which is greater than it. One gets that the first 3 numbers are 452. One put the smallest number in place of 7 that is not in the arrangement, that is «1». As a result one gets the arrangement 4521 which is lexicographically next after 4517.

Let's repeat the previous actions: the last number, which can be increased is «1», which one increases on the one of the following numbers {3, 6, 7}, namely, «3», and one gets 4523 as a result. In the same way one can get the following numbers: 4526 and 4527.

Since the last digit in the arrangement «7» can not be increased then one can increase the following number «2» on the smallest which is simultaniously greater than «2», the rest of the numbers can be ordered in the ascending order (the numbers from the set), one gets the next arrangement 4531.

If one continues the process, one gets 4532, 4536, 4537, 4561, 4562.

Task 2. The set {1, 2, 3, 4, 5, 6, 7} is given. Build 5 combinations without repetitions from 4 elements that are lexicographically next after {1, 4, 5, 6}.

Solution.

The combination can be written without parentheses for the sake of simplicity, therefore, {1, 4, 5, 6} can be represented as 1456, bearing in mind that the numbers can be only in ascending order.

Let's build the next lexicographic combination after 1456. It is obvious that it can be obtained by increasing the last element, and one gets 1457.

After the obtaining of next lexicographic combination after 1457, the last number cannot be increased (since 7 is the largest number in the set with the elements in the ascending order). Therefore, one increases the second digit from the right, namely, «5». One gets the first three numbers of the permutation146. Since the order of numbers in the permutation should be ascending then one can put the number on the last place that is greater than «6», then the next permutation is 1467.

Let’s build the next lexicographic combination after the 1467. The last number cannot be increased then one moves to «6». The number «6» cannot be increased as well, since one gets the first three numbers of the permutation equal to 147, and then the last number has to be greater than «7». Then one moves to the next number «4». It can be increased to «5», and the next numbers should be in the ascending order. From this follows that the next permutation should be only 1567.

Using the previous considerations one can increase only the last element in the next combination, therefore, the first element of permutation will be «2». The next numbers are chosen in the ascending order б, and one gets 2345.

Task 3. One needs to choose 7 developers from the twenty to work on the software project. In how many ways this can be done? How many ways there exists to build a list of seven developers from twenty?

How many ways are there to assign 7 different tasks to the developers? How many ways are there to split the responsibility between the developers for the support of the project work, so that three are responsible for Unit tests, two are responsible for the continuous project integration and two for the delivery of intermediate versions to the client?

It is known that the project manager sent 5 different emails with instructions to certain developers as a result of project work. In how many ways this can be done if neither addressee gets more than one email? In how many ways this can be done if each addressee can get more than one letter?

Solution.

The sample of a group of seven developers from the twenty does not have repetitions and order, therefore, it is a combination without repetitions and the number of its ways is equal to:

$$C_{20}^{7} = \frac{20!}{13! \cdot 7!} = 77520.$$

If one arranges the list of developers, than the sample will be ordered, therefore, one can use the formula of arrangements without repetitions:

$$A_{20}^{7} = \frac{20!}{13!} = 390700800.$$

In order to calculate the number of task assignments, let’s imagine that one has the fixed list of developers and one writes its task near each name. Changing the order of the tasks whilst remaining the order of developers’ names fixed, one gets various variants of tasks assignments. Therefore, the samples are ordered, they do not have repetitions, and each of which contains all elements. Thus, this is the permutation without repetitions.

$$P_7 = 7! = 5040$$

In order to get the number of distribution of additional roles, let’s use the previous considerations. Therefore, the single difference with the previous task is rhat this variant deals with permutation with repetitions:

$$P_7(3,2,2) = \frac{7!}{3!\cdot 2!\cdot 2!} = 210.$$

Let’s calculate the number of ways to send the emails. In this case the sample is an arrangement of five persons from all seven project developers, In the case, is the email would be the same, we would have a combination, but here one sends the different emails to different persons, therefore, the sample is an arrangement. The sample cannot be also the permutation, since the emails were sent to only five persons. In the first case all addressee are different, therefore the arrangement does not have repetitions:

$$A_7^5 = \frac{7!}{2!} = 2520.$$

In the second case the addressee can repeat, so one can use the formula of arrangement with repetitions:

$$\tilde{A}_7^5 = 7^5 = 16807.$$

Task 4. In how many ways one can choose two pairs of cards (two queens and two aces, two twos and two jacks, тощо)? The order of cards is not important.

Solution.

In order to calculate the number of ways, one builds the sequence of action, which can lead to two pairs.

1) Let’s choose the rank of the card (ace, two, three and so on) for two pairs, namely, one determines whether these will be aces and threes, queens and jacks, or

other pairs. On this step we have not chosen the suits of the cards.

2) Then one chooses the suits for one pair.
3) Then the suit is chosen for the other pair.

All action are made one after another, then one can find the number of ways to perform each action, and, using the product rule, multiply the obtained numbers.

Let's calculate the number of ways to perform the first action. There are 13 ranks in a deck of 52 cards (ace, two, three and so on). The chosen ranks should be different, since one would get four cards of the same rank instead of two pairs. Also, the order of cards is not important, therefore, one should use the formula for combinations without repetitions. One gets two ranks from 13, and then the first action can be done in C_{13}^2 ways.

Let's calculate the number of ways to perform the second action. There are 4 suits in the cards and there are no duplicates between the cards. Thus, the chosen suits will be different (without repetitions). The order of cards is not important, therefore, one should employ the formula for the combinations without repetitions. One chooses 2 suits from 4, then the second action can be done in C_4^2 ways.

Similarly, the third action can be done in C_4^2 ways.

Therefore, all actions according to the product rule can be done in

$$C_{13}^2 \cdot C_4^2 \cdot C_4^2 = \frac{13 \cdot 12}{2 \cdot 1} \cdot \frac{4 \cdot 3}{2 \cdot 1} \cdot \frac{4 \cdot 3}{2 \cdot 1} = 2808 \text{ ways.}$$

Task 5. How many different strings can be obtained from the word «misspelled», using all letters? How many such strings start from the letter «i»?

Solution.

Let's determine the type of the sample which suits for this task. The letters in the new strings are ordered, therefore, their various arrangement gives different strings. Also, the strings will have the repetitions, though the repetition of the letter «s» will happen exactly twice, as well as the letters «e» and «l». Therefore, it is not the arrangement with repetitions, but the permutation with repetitions.

Therefore, the number of such strings is equal to

$$\mathrm{P}(2,2,2,1,1,1,1) = \frac{10!}{2! \cdot 2! \cdot 2! \cdot 1! \cdot 1! \cdot 1! \cdot 1!} == \frac{10 \cdot 9 \cdot 8 \cdot 7 \ldots \cdot 1}{2 \cdot 2 \cdot 2} = 453600$$

If the letter «i» is on the first place, then we don't consider it in the permutation, therefore, the number of such permutations is

$$P(2,2,2,1,1,1) = \frac{9!}{2!\cdot 2!\cdot 2!\cdot\cdot 1!\cdot 1!\cdot 1!} == \frac{9\cdot 8\cdot 7\ldots\cdot 1}{2\cdot 2\cdot 2} = 45360$$

Task 6. In the expansion $\left(\sqrt[3]{a}+\sqrt{a^{-5}}\right)^{34}$ find the term which does not depend on a.

Solution.

Let's use the Newton's binomial theorem:

$$\left(\sqrt[3]{a}+\sqrt{a^{-5}}\right)^{34} = \sum_{k=0}^{34} C_n^k \left(\sqrt[3]{a}\right)^k \left(\sqrt{a^{-5}}\right)^{34-k}$$

In the term of Newton binomial expansion, that does not depend on a, the sum of powers of parameter a should cancel out and should be equal to zero.

$$\left(\sqrt[3]{a}\right)^k \left(\sqrt{a^{-5}}\right)^{34-k} = a^{\frac{k}{3}} a^{\frac{-5(34-k)}{2}} = a^0 = 1$$

From this one can find the sought expansion term k. Let's write down the identity for powers:

$$\frac{k}{3} + \frac{-5(34-k)}{2} = 0,$$

From this one gets

$$\frac{2k - 15(34-k)}{6} = 0$$

$$17k = 510;$$

$$k=30.$$

Therefore, if one assumes that $k = 0$ in the binomial expansion determines the first term, when $k = 1$ one gets the second term, and so on, then when $k = 30$ one finds the 31st expansion term.

Let's calculate it:

$$C_{34}^{30} a^{\frac{30}{3}} a^{\frac{-5(34-4)}{2}} = C_{34}^{30} a^{10-10} = C_{34}^{30}$$

The sought expansion term is 31st and is equal to C_{34}^{30}.

Task 7. In the expansion $(x+y)^n$ the coefficient of the fourth term is equal to 969. Find *n*.

Solution.

If one assumes that when *k* = 0 one gets the first expansion term, when *k* = 1 one gets the second term and so on, then the coefficient of the fourth term is equal to

$$C_n^3 = \frac{n!}{3!(n-3)!} = \frac{(n-3)!(n-2)(n-1)n}{6(n-3)!} = \frac{(n-2)(n-1)n}{6}.$$

Since it is equal to 969, then:

$$(n-2)(n-1)n = 969 \cdot 6 = 5814.$$

In order to avoid solving cubic equation $n^3 + 3n^2 + 2n - 5814 = 0$,
one can find three consecutive natural numbers, which product is equal to 5814. One possible option is factorization of 5814. But the simplest way is to find the cubic root of the number 5814, to find approximately *n*, and then using the direct search method to find it.

$$\sqrt[3]{5814} \approx 17{,}9814.$$

Searching the variants *n* = 19, 18, 17, one gets that:

$$(n-2)(n-1)n = 17 \cdot 18 \cdot 19 = 5814$$

Therefore, *n* = 19.

Task 8. How many persons there should be so that at least four of them were born at the same day of the week and at the same day of month?

Solution.

To find the number of variants, lets multiple the number of days in the week (7) on the number of days in the month (31), then one gets 217, that is the number of possible birthdays in various days of week and day of month.

Thus, while gathering 217 persons, one cannot guarantee, that between them two of them were born on the same day of the week and on the same day of the month. If the number of such people is 218, then at least two persons are in this group.

Similarly, in the worst case, when gathering $217 \cdot 3 = 651$ persons one cannot guarantee, that at least 4 people were born on the same day of the week and on the same day of the month. According to pigeonhole principle, if one invites one more person then there will be 4 such people. Therefore, there should be at least 652 persons.

4.2. Tasks for self-study

Task 1. Let M={1,2,3,4,5}. List all the arrangements and combinations without repetitions from the elements of the set M of 3 elements.

Task 2. Arrange all the given permutations of the elements of the set {1, 2, 3, 4, 5, 6} in lexicographic order: 456321, 231564, 132456, 156423, 165432, 543216, 541236, 314562, 341526, 654312, 432561, 653412.

Task 3. Find the lexicographically next for each of the permutations: 54123; 1432; 12453; 45231; 6714235; 31528764.

Task 4. Using the algorithm of construction of the lexicographically next permutation, write down the first 10 arrangements from 6 of 4 elements of the set {1, 2, 3, 4, 5, 6}.

Task 5. List all the arrangements without repetitions of two elements and arrangement with repetitions of two elements of the set {1, 2, 3, 4}. List all the combinations with repetitions of two elements from the same set.

Task 6. List all the combinations without repetition of three elements of the set {1, 2, 3, 4, 5}.

Task 7. List all the combinations with repetition of three elements of the set {1, 2, 3, 4, 5}.

Task 8. Calculate the number of permutations of the set {a, b, c, d, e}, that ends with the letter d and starts with the letter b.

Task 9. In how many ways can the prize places (the first, the second, and the third) be determined in a run of 10 horses?

Task 10. In how many ways one can host 12 students in 4 rooms of the hostel, if each room should contain 3 guests?

Task 11. There is a group of n male and n female. In how many ways can they be arranged in a row, so that the male and female were one after another?

Task 12. Suppose a set M={1, 2, 3, …, 19, 20is given. How many arrangements without repetitions exist from the elements of set M of four elements which contain:

a) the number 13;

b) the numbers 13 and 14, simultaneously;

c) the numbers 13, 14 i 17, simultaneously;

d) the numbers 13, 14, 17 i 20, simultaneously;

e) four consecutive numbers in the ascending order;

f) three consecutive numbers in the ascending order?

Task 13. How many ways are there to place five persons at the round table?

Task 14. In how many ways can one form a pair of tiles from 28 dominoes that can be placed next to each other according to the rules of dominoes?

Task 15. In how many ways can five men and five women be seated at a round table so that two men do not sit next to each other?

Task 16. From the numbers 1, 2, 3, 4, 5, without repeating them, make all five-digit numbers. How many numbers among them are, that:

a) begin with the number 3;

b) do not begin with the number 5;

c) start with 54?

Task 17. Natural numbers from 1 to 25 are given. In how many ways can one choose two numbers from them so that their sum is an even number? Three numbers? Four?

Task 18. In how many ways can one put 9 books on the shelf:

a) if there is one three-volume book among them, all the volumes of which must stand next to each other in random order;

b) so that all volumes of the three-volume set are next to each other in ascending order of volume numbers?

Task 19. How many participants are in a chess tournament, if it is known that each participant played with each of the others, and a total of 210 games took place?

Task 20. In how many ways can 10 cards be removed from a deck of 52 cards so that among them are the following:

a) exactly one ace;

b) at least one ace;

c) at least two aces?

Task 21. In how many ways can one choose a pair of identical cards (two queens, two aces, etc.) from a deck of 36 cards? Three identical cards? Three cards of the same suit?

Task 22. How many different strings of five letters can be formed from an alphabet that has 26 letters if repetitions are allowed?

Task 23. How many different strings can be formed from the word MISSISSIPPI using all the letters? How many such strings begin and end with the letter S? In how many such strings are all four letters S next to each other?

Task 24. The set contains 10 elements. Find the number of subsets of this set that contain more than one element.

Task 25. How many bit strings can be formed from seven zeros and four ones?

Task 26. How many bit strings can be formed from eleven zeros and three ones, if each string must start with a one and each one must be followed by at least two zeros?

Task 27. Build an expansion:

$a)\ (x+y)^5$; $b)\ (x-y)^5$; $c)\ (x+y)^6$; $d)\ (x-y)^6$.

Task 28. Determine the coefficients in the expansion $(x-y)^9$ near x^2y^7, x^4y^5, x^7y^2.

Task 29. Determine the fifth term of the binomial expansion $\left(\frac{a}{\sqrt{x}}+\frac{\sqrt{x}}{a}\right)^n$, if the ratio of the coefficient of the third term to the coefficient of the second term is equal to 11/2. The terms of the binomial are numbered from 1 to n + 1.

Task 30. In the expansion of the binomial $(\sqrt{1+x}-\sqrt{1-x})^n$, the coefficient of the third term is equal to 28. Determine the middle term of the expansion.

Task 31. Determine the smallest value of the power n in the expansion $(1+x)^n$ for which the ratio of two adjacent coefficients is equal to 7/15.

Task 32. In the expansion of the binomial $(\sqrt[3]{a}+\sqrt{a^{-1}})^{15}$, determine the term that does not depend on a.

Task 33. How many rational members are contained in the expansion $(\sqrt{2}+\sqrt[4]{3})^{100}$?

Task 34. In the expansion of the binomial $\left(a\sqrt[5]{a/3}-b/\sqrt[7]{a^3}\right)^n$, determine the term containing a^3, if the sum of the binomial coefficients in odd places in the expansion is equal to 2048.

Task 35. For what value of n, the coefficients of the second, third, and fourth members of the binomial expansion $(x+y)^n$ form an arithmetic progression?

Task 36. Let M be a finite set. Prove that there are as many subsets of the set M with an even number of elements as there are subsets with an odd number of elements.

Task 37. Prove the binomial theorem algebraically using mathematical induction.

Task 38. Prove that $C_n^r = P(r, n-r)$.

Task 39. Write down the expansion $(x+y+z)^4$.

Task 40. Find the coefficient near $x^3y^2z^5$ in the expansion $(x+y+z)^{10}$.

Task 41. Find the number of terms in the expansion $(x_1 + x_2 + ... + x_k)^n$.

Task 42. How many people should be invited so that at least six of them were born under the same zodiac sign?

Task 43. How many people must there be so that at least two of them were born on the same day of the week and in the same month (perhaps in different years)?

Task 44. Let M be the set of ten natural numbers that do not exceed 50. Prove that there are at least two different five-element subsets of the set M such that the sums of their elements are equal.

Task 45. How many elements does the union of five sets contain, if each of them contains 10,000 elements, each pair has 1,000 common elements, each three has 100 common elements, each four has 10 common elements, and one element belongs to all five sets?

Task 46. How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, if x_1, x_2, x_3 are non-negative integers smaller than 6?

Task 47. Find the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 = 17$, if x_1, x_2, x_3, x_4 are integers such that $x_1 \le 3$, $x_2 \le 4$, $x_3 \le 5$, $x_4 \le 8$.

Task 48. Find the number of solutions of the following equations in non-negative integers:

a) $x_1 + x_2 + x_3 = 15$;

b) $x_1 + x_2 + x_3 + x_4 = 17$;

c) $x_1 + x_2 + x_3 = 15$ when $x_1 \ge 2, x_2 \ge 4, x_3 \ge 5$.

Task 49. Let x_1, x_2, x_3, x_4, x_5 be non-negative integers. Find the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 14$ under the following conditions:

a) $x_1 \ge 1$;

b) $x_j \ge 2$ for j = 1, 2, 3, 4, 5;

c) $0 \le x_1 \le 10$;

d) $0 \le x_1 \le 3, 1 \le x_2 \le 4, x_3 \ge 5$.

Task 50. Find the number of solutions of the inequality $x_1 + x_2 + x_3 \le 15$ in non-negative integers.

Task 51. Find the number of positive integers less than 1,000,000 whose sum of digits is 19.

Task 52. Find the number of positive integers less than 1,000,000 that have an exactly one digit 9, and the sum of their digits is 13.

Task 53. From the numbers 1, 2, 3, 4, 5, without repeating them, make all five-digit numbers. How many numbers among them:

a) begin with the number 3;

b) do not begin with the number 5;

c) start with 54?

Task 54. In how many ways one can put 9 books on a shelf if there is one three-volume book among them, all the volumes of which should stand next to each other in random order;

Task 55. In how many ways one can arrange 28 different objects in four different boxes, so that there are 7 objects in each box?

Task 56. In how many ways can you put 28 different postcards in 4 identical envelopes so that there are 7 postcards in each envelope?

Task 57. An elevator with 9 passengers can stop on ten floors. Passengers leave in groups of two, three and four. In how many ways can this happen?

Task 58. Twelve students were given two test options. In how many ways can they be seated in two rows so that there are no identical options next to each other, and those who sit one behind the other have the same option?

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ДЛЯ НОТАТОК

НАВЧАЛЬНЕ ВИДАННЯ

МЕТОДИЧНІ ВКАЗІВКИ

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з дисципліни
«ДИСКРЕТНА МАТЕМАТИКА»

галузі знань

12 «Інформаційні технології»

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