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Department of Mathematical Methods in Engineering

Educational and methodical manual

for self study of students of all forms of studies

with the

“Elements of Vector Algebra”

of Higher Mathematics course

Ternopil
2020

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H15

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Contents

1. Scalars and vectors. Types of vectors

Many physical quantities such as mass, distance, length, volume, temperature can be specified completely by giving a single number. This type of a physical quantity is called a **scalar quantity**. Scalar is a synonym of “number.”

However, there are a lot of physical quantities: velocity, force, torque, which cannot be described completely by just a single number of physical units. Those quantities have both a number of units (magnitude) and a direction to specify completely. Such quantities are known as **vector quantities** or **vectors**.

The general definitions of scalars and vectors are given below.

Definition. The quantity which is defined by a single number with appropriate units is called a **scalar quantity** or simply **scalar**.

Definition. The quantity which is defined completely by giving a number of units and a direction is called a **vector quantity or vector**.

Geometrically, we can represent a vector as a directed line segment, an “arrow” with initial point A (tail) and terminal point B (head) as shown on Figure 1.1. The arrow’s length is the magnitude of the vector. The direction of the vector is indicated by an arrow pointing from the tail to the head.

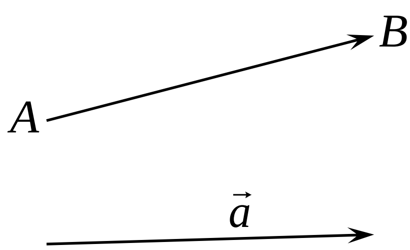


Figure 1.1

The vector is denoted by one of these symbols $\overrightarrow{AB}$ or $\vec{a}$.

When denoting a vector by two capital letters the first one defines the initial point of vector, and the second one defines its terminal point.

The magnitude of vector is denoted as $|\overrightarrow{AB}|$ or $|\vec{a}|$.

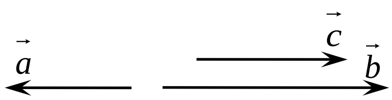
Definition. The vector, for which the magnitude and direction are defined, though the point of application is not provided, is called a **free vector**.

The concept of free vector permits its parallel translation (movement) in any point of the space with the preserving of its length and direction.

Definition. A vector whose initial and terminal points are the same is called a **null vector** or **zero vector**. Its magnitude is zero and its direction is indeterminate. It is denoted as $\vec{0}$.

Definition. The vector whose magnitude is equal to one unit is called a **unit vector** or **normalized vector**.

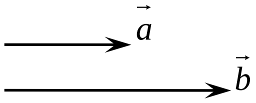
Definition. The vectors $\vec{a}$ and $\vec{b}$ are called **collinear**, if they lie on the same line or parallel lines (see Figure 1.2). This fact is denoted as: $\vec{a} \parallel \vec{b}$.



$$\vec{a} \parallel \vec{b} \parallel \vec{c}$$

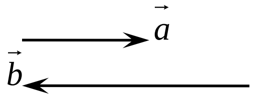
Figure 1.2

One distinguishes co-directed and oppositely directed vectors. They are denoted as $\vec{a}\uparrow\uparrow\vec{b}$ (Figure 1.3) or $\vec{a}\uparrow\downarrow\vec{b}$ (Figure 1.4), respectively.



$\vec{a}\uparrow\uparrow\vec{b}$

Figure 1.3



$\vec{a}\uparrow\downarrow\vec{b}$

Figure 1.4

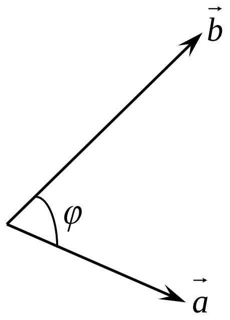


Figure 1.5

To determine the angle between two vectors, one needs to translate them to the common initial point (Figure 1.5).

- If $\vec{a}\uparrow\uparrow\vec{b}$ then $\varphi = 0^\circ$.
- If $\vec{a}\uparrow\downarrow\vec{b}$ then $\varphi = 180^\circ$.

Definition. The opposite directed vectors of the same length are called **opposite vectors**.

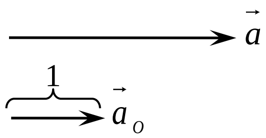


Figure 1.6

Definition. The unit vector, which is co-directed with the vector $\vec{a}$ is called a **normalized vector (**or **versor** or **direction vector)** of vector $\vec{a}$ (Figure 1.6).

The normalized vector $\vec{a}_0$ of vector $\vec{a}$ can be found by dividing $\vec{a}$ by its length $|\vec{a}|$; that is,

$$\vec{a}_0 = \frac{\vec{a}}{|\vec{a}|}. \qquad (1.1)$$

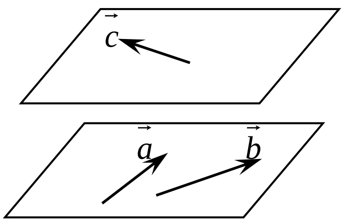


Figure 1.7

The direction vectors of axes *OX*, *OY*, *OZ* of rectangular Cartesian coordinate system are denoted as $\vec{i}$, $\vec{j}$, $\vec{k}$, respectively:

$$|\vec{i}| = |\vec{j}| = |\vec{k}| = 1, \quad \vec{i}\perp\vec{j}, \quad \vec{j}\perp\vec{k}, \quad \vec{i}\perp\vec{k}.$$

Definition. The vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are called **coplanar** if they lie on the same plane or parallel planes (Figure 1.7).

Questions for self-check

1) Formulate a definition of a scalar quantity.
2) Formulate a definition of a vector quantity.
3) When denoting vector by two capital Latin letters the first letter determines …
4) What is the geometrical meaning of vector's magnitude?
5) The vector, which magnitude and direction are defined, but the point of application is not given, is called...
6) Give the definition of a zero vector.
7) What is the magnitude of a zero vector?
8) Give the definition of the unit vector.
9) The vectors, which lie on the same line or on the parallel lines, are called...
10) To determine the angle between the vectors, it is necessary to …
11) The angle between the co-directed vectors is equal to …
12) The angle between oppositely directed vectors is equal to …
13) Give the definition of oppositely directed vectors.
14) The unit vector, which is co-directed with the vector $\vec{a}$, is called…
15) Which vectors are called coplanar?
16) If $\vec{u}$ and $\vec{v}$ are nonzero vectors with $|\vec{u}|=|\vec{v}|$, does it follow that $\vec{u}=\vec{v}$?
17) What are vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$ called?

2. The projection of vector on the axis. The components of the vector. The direction cosines of the vector

2.1. The projection of a vector on the axis

Let the vector $\vec{a}$ and the axis l are given. The angle between the vector $\vec{a}$ and a positive direction of axis l is equal to φ (Figure 2.1).

Definition. The product of length of vector $\vec{a}$ on the cosine of angle φ between the vector and the axis is called the **projection of vector $\vec{a}$ on the axis** *l*.

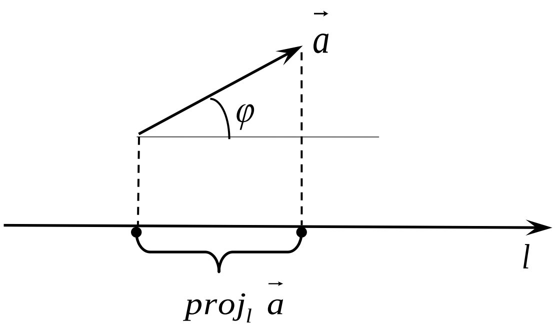


Figure 2.1

The projection of vector $\vec{a}$ on the axis l can be found by the formula

$$proj_l\,\vec{a} = |\vec{a}| \cdot \cos\varphi. \qquad (2.1)$$

- If $0 < \varphi < 90^\circ$, then $proj_l\,\vec{a} > 0$.
- If $90^\circ < \varphi < 180^\circ$, then $proj_l\,\vec{a} < 0$.

2.2. The vector on the plane *OXY*

Let vector $\vec{a}$ makes the angles α and β with the positive x and y coordinate directions respectively: $\angle(\vec{a},\vec{i}) = \alpha$, $\angle(\vec{a},\vec{j}) = \beta$.

Definition. The cosines of angles, which the vector $\vec{a}$ makes with the positive coordinate axes, are called the **direction cosines** of vector $\vec{a}$.

The sum of squares of direction cosines is equal to 1:

$$\cos^2\alpha + \cos^2\beta = 1. \qquad (2.2)$$

Definition. The projections of vector $\vec{a}$ on the axes *OX*, *OY* of the rectangular coordinate system are called the **coordinates of vector (or the components)** in this system of coordinates. The projections of vector $\vec{a}$ on these coordinate axes (Figure 2.2) are denoted as $proj_{OX}\,\vec{a}$, $proj_{OY}\,\vec{a}$ or a_x, a_y respectively and is calculated by the formulas:

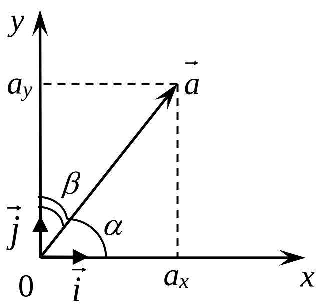


Figure 2.2

$$a_x = |\vec{a}| \cdot \cos\alpha, \qquad a_y = |\vec{a}| \cdot \cos\beta. \qquad (2.3)$$

Using the formulas (2.3), one gets the formulas that allow to determine the direction cosines:

$$\cos\alpha = \frac{a_x}{|\vec{a}|}, \qquad \cos\beta = \frac{a_y}{|\vec{a}|} \tag{2.4}$$

Therefore, the direction cosines of a vector $\vec{a}$ are the coordinates of the normalized vector of this vector:

$$\vec{a}_0 = (\cos\alpha, \cos\beta).$$

Definition. The **vector** in the two-dimensional rectangular coordinate system is an ordered pair of numbers a_x, a_y: $\vec{a} = (a_x, a_y)$.

The magnitude of vector $\vec{a} = (a_x, a_y)$ can be found by Pythagorean theorem:

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2}. \tag{2.5}$$

For the vector $\overrightarrow{AB}$, that is defined by the coordinates of the initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$, its projections on the coordinate axes will be:

$$a_x = x_2 - x_1, \qquad a_y = y_2 - y_1.$$

Therefore, ***to find the components of the vector, it is necessary to subtract the coordinates of the initial point of vector from the respective coordinates of the terminal point of vector***

$$\overrightarrow{AB} = (x_2 - x_1,\ y_2 - y_1). \tag{2.6}$$

The magnitude of the vector $\overrightarrow{AB}$ can be found by the formula:

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{2.7}$$

2.3. The vector in the space *OXYZ*

Let vector $\vec{a}$ is placed in the rectangular coordinate system *OXYZ*, and $\angle(\vec{a}, \vec{i}) = \alpha$, $\angle(\vec{a}, \vec{j}) = \beta$, $\angle(\vec{a}, \vec{k}) = \gamma$ (Figure 2.3). Then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of vector $\vec{a}$ in a rectangular coordinate system *OXYZ*, and

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \tag{2.8}$$

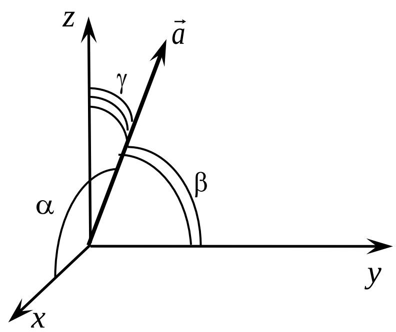


Figure 2.3

The direction cosines of vector $\vec{a}$ are the coordinates of normalized vector of this vector:

$$\vec{a}_0 = (\cos\alpha, \cos\beta, \cos\gamma).$$

The coordinates of vector $\vec{a} = (a_x, a_y, a_z)$ are determined by the formulas:

$$a_x = \left|\vec{a}\right| \cdot \cos\alpha, \quad a_y = \left|\vec{a}\right| \cdot \cos\beta, \quad a_z = \left|\vec{a}\right| \cdot \cos\gamma. \qquad (2.9)$$

Definition. The **vector** in three-dimensional rectangular coordinate system is an ordered triple of numbers a_x, a_y, a_z.

The magnitude of vector $\vec{a} = \left(a_x, a_y, a_z\right)$ can be found by the formula:

$$\left|\vec{a}\right| = \sqrt{a_x^2 + a_y^2 + a_z^2}. \qquad (2.10)$$

The coordinates of vector $\overrightarrow{AB}$, that is given by the coordinates of initial point $A\left(x_1, y_1, z_1\right)$ and the terminal point $B\left(x_2, y_2, z_2\right)$, can be found by formulas:

$$a_x = x_2 - x_1, \;\; a_y = y_2 - y_1, \;\; a_z = z_2 - z_1, \qquad (2.11)$$

Therefore,

$$\overrightarrow{AB} = \left(x_2 - x_1, \; y_2 - y_1, \; z_2 - z_1\right), \qquad (2.12)$$

Then the magnitude of vector $\overrightarrow{AB}$ is calculated by the formula:

$$\left|\overrightarrow{AB}\right| = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}. \qquad (2.13)$$

Questions for self-check

1) The product of the length of vector $\vec{a}$ on the cosine of angle φ between the vector and the axis is called…
2) The projection of vector $\vec{a}$ on the axis *l* can be found by the formula…
3) Which sign has the projection of vector on the axis, if the angle between them is acute?
4) Which sign has the projection of vector on the axis, if the angle between them is obtuse?
5) Provide the definition of direction cosines of vector.
6) Write down the property of direction cosines of the vector.
7) The projection of the vector $\vec{a}$ on the axis *OX* is denoted as …
8) The projection of the vector $\vec{a}$ on the axis *OY* can be found by the formula ...
9) The magnitude of vector $\vec{a} = \left(a_x, a_y\right)$ can be found by the formula ...
10) To find the coordinates of the vector having the coordinates of its initial and terminal point, it is necessary to …

Solved problems

1. Find the projection of vector $\vec{a}$ on the axis *l*, if $|\vec{a}| = 2$, and the angle between the vector and the axis is equal to 45°.

Solution. $np_l\,\vec{a} = |\vec{a}| \cdot \cos\varphi = 2 \cdot \cos 45^{\circ} = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$.

Answer. $\sqrt{2}$.

2. Given points $A(3;-1)$ and $B(-1;2)$, find the coordinates of vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$.

Solution. $\overrightarrow{AB} = (-1-3; 2-(-1)) = (-4;3)$, $\overrightarrow{BA} = (3-(-1); -1-2) = (4;-3)$.

Answer. $\overrightarrow{AB} = (-4;3)$, $\overrightarrow{BA} = (4;-3)$.

3. Determine the point *N*, which coincides with the terminal point of the vector $\vec{a} = (3;-1;4)$, if its initial point coincides with the point $M(1;-1;2)$.

Solution. Let $N(x;y;z)$. Since $\vec{a} = \overrightarrow{MN}$, then $(3;-1;4) = (x-1; y+1; z-2)$, and

$$\begin{cases} x-1=3 \\ y+1=-1 \\ z-2=4 \end{cases} \Rightarrow \begin{cases} x=4 \\ y=-2 \\ z=6 \end{cases} \Rightarrow N(4;-2;6).$$

Answer. $N(4;-2;6)$.

4. Determine the direction cosines of the vector $\vec{a} = (3;-4)$.

Solution. By the formulas (2.4) $\cos\alpha = \frac{a_x}{|\vec{a}|}$, $\cos\beta = \frac{a_y}{|\vec{a}|}$. Since $a_x = 3$, $a_y = -4$,

then $|\vec{a}| = \sqrt{3^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$, $\cos\alpha = \frac{3}{5}, \cos\beta = -\frac{4}{5}$.

Answer. $\cos\alpha = \frac{3}{5}, \cos\beta = -\frac{4}{5}$.

5. Can the vector form the following angles with the coordinate axes?

1) $\alpha = 45^{\circ}, \beta = 60^{\circ}, \gamma = 120^{\circ}$; 2) $\alpha = 45^{\circ}, \beta = 135^{\circ}, \gamma = 60^{\circ}$.

Solution. The direction cosines of vector should satisfy the identity $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. Let's check this identity in each case.

$$\cos^2 45° + \cos^2 60° + \cos^2 120° = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1.$$

The identity holds true, therefore, the vector can form the given angles with coordinate axes.

$$\cos^2 45° + \cos^2 135° + \cos^2 60° = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} + \frac{3}{4} = \frac{5}{4} \neq 1.$$

The identity does not hold true, therefore, the vector cannot form the given angles with the coordinate axes.
Answer. 1) Yes; 2) No.

6. Calculate the magnitude of vector $\vec{a} = (6;3;-2)$.

Solution. $|\vec{a}| = \sqrt{6^2 + 3^2 + (-2)^2} = \sqrt{36+9+4} = \sqrt{49} = 7$.

Answer. 7.

7. Find the coordinates of vector $\vec{a}$, if there are given its magnitude and the angles α, β, γ, which form the vector $\vec{a}$ with the coordinate axes: $|\vec{a}| = 8$, $\alpha = \pi/3$, $\beta = \pi/3$.

Solution. The coordinates of vector $\vec{a} = (a_x; a_y; a_z)$ can be determined by the formulas (2.9), where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of vector $\vec{a}$. The unknown is $\cos\gamma$. Since the direction cosines of the arbitrary nonzero vector $\vec{a}$ are connected with the relation (2.8), then $\cos\gamma = \pm\sqrt{1 - \cos^2\alpha - \cos^2\beta}$.

$$\cos\gamma = \pm\sqrt{1 - \cos^2\frac{\pi}{3} - \cos^2\frac{\pi}{3}} = \pm\sqrt{1 - \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} = \pm\sqrt{1 - \left(\frac{1}{4}\right) - \left(\frac{1}{4}\right)} = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}.$$

Therefore, the coordinates of vector

$$a_x = 8\cos\frac{\pi}{3} = 8 \cdot \frac{1}{2} = 4, \quad a_y = 8\cos\frac{\pi}{3} = 8 \cdot \frac{1}{2} = 4, \quad a_z = \pm 8\frac{\sqrt{2}}{2} = \pm 4\sqrt{2}.$$

The vectors are: $\vec{a}_1 = (4;4;-4\sqrt{2})$ or $\vec{a}_2 = (4;4;4\sqrt{2})$.

Solution. $\vec{a}_1 = (4;4;-4\sqrt{2})$ or $\vec{a}_2 = (4;4;4\sqrt{2})$.

3. The equality of vectors. The linear operations on vectors

Definition. Two vectors $\vec{a}$ and $\vec{b}$ are called **equal**, if they are co-directed and have the same magnitude.

If the coordinates of vectors are known $\vec{a} = \left(a_x, a_y, a_z\right)$ and $\vec{b} = \left(b_x, b_y, b_z\right)$, then the equality $\vec{a} = \vec{b}$ is equivalent to the system

$$\begin{cases} a_x = b_x, \\ a_y = b_y, \\ a_z = b_z. \end{cases} \qquad (3.1)$$

The vectors are equal, if all the respective coordinates of these vectors are equal.

The linear operations on vectors are the multiplication of vector on a number, the sum of vectors, and the difference of vectors.

<u>The multiplication of vector on a number.</u>

The result of multiplication of vector $\vec{a}$ on a number k is a vector which has a magnitude $k\left|\vec{a}\right|$, and it is co-directed with $\vec{a}$, if $k > 0$; and oppositely directed with $\vec{a}$, if $k < 0$ (Fig. 1).

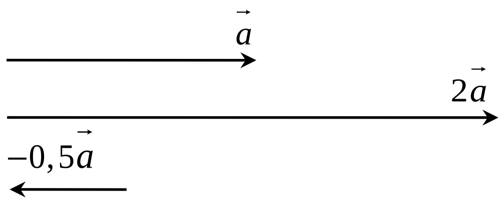


Figure 3.1.

The vectors $\vec{a}$ and $\left(-1\right)\cdot\vec{a} = -\vec{a}$ are opposite, the vector $\vec{a}$ and $k\vec{a}$ are collinear.

If the coordinates of vector $\vec{a} = \left(a_x, a_y, a_z\right)$ are given, then

$$k\vec{a} = \left(ka_x, ka_y, ka_z\right) \qquad (3.2)$$

To multiply the vector on the number, it is necessary to multiply all coordinates of vector on that number.

<u>The sum of vectors.</u> **Geometrically** two vectors are added using the triangle rule (the terminal point of the first vector coincides with the initial point of the first vector) or by the parallelogram rule (the vectors have the same initial point), that is shown on Fig. 2.

Figure 3.2

If the components of vectors are known $\vec{a}=\left(a_x,a_y,a_z\right)$ and $\vec{b}=\left(b_x,b_y,b_z\right)$, then their sum

$$\vec{a}+\vec{b}=\left(a_x+b_x,\ a_y+b_y,\ a_z+b_z\right) \tag{3.3}$$

To add two vectors, it is necessary to add all the respective components.

<u>The subtraction of vectors.</u> ***Geometrically*** two vectors that have the same initial point, are subtracted using the rule, depicted on Fig. 3.

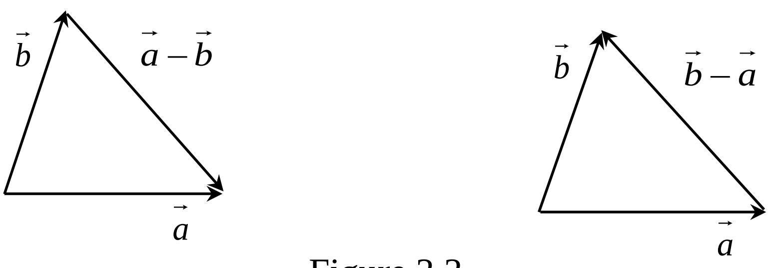


Figure 3.3

If the components of vectors are known $\vec{a}=\left(a_x,a_y,a_z\right)$ and $\vec{b}=\left(b_x,b_y,b_z\right)$, then their difference

$$\vec{a}-\vec{b}=\left(a_x-b_x,\ a_y-b_y,\ a_z-b_z\right) \tag{3.4}$$

To subtract two vectors, it is necessary to subtract the respective components.

Questions for self-check

1) If two vectors are co-directed and have the same magnitude, then they are called …
2) Write down the equality of vectors in coordinate form.
3) Which operations on vectors are called the linear operations?
4) When multiplying the vector $\vec{a}$ on the number k one gets the vector, which has the magnitude...
5) What is the relation between the vectors $\vec{a}$ and $-\vec{a}$?
6) When multiplying the vector on the positive number its direction…
7) When multiplying the vector on the negative number its direction …
8) To multiply the number on the vector, given by its components, it is necessary…
9) The geometrical addition of vectors is performed by the rule …
10) To add two vectors in the component form, it is necessary …
11) To subtract two vectors in the component form, it is necessary …

Solved problems

1. Given two vectors: $\vec{a}=(4;-1;7)$ and $\vec{b}=(-1;2;1)$. Find the projections of vector $\vec{a}-\vec{b}$ on the coordinate axes.

Solution. $\vec{a}-\vec{b}=(4;-1;7)-(-1;2;1)=(5;-3;6)$. The components of the obtained vector are the projections of this vector on the coordinate axes.

Answer. $5;-3;6$.

2. Find the versor of vector $\vec{a}=(6;-2;-3)$.

Solution. Using the formula (1.1): $\vec{a}_0=\dfrac{\vec{a}}{|\vec{a}|}$. Let's calculate the magnitude of vector: $|\vec{a}|=\sqrt{6^2+(-2)^2+(-3)^2}=\sqrt{36+4+9}=7$.

Then $\vec{a}_0=\dfrac{1}{7}\cdot(6;-2;-3)=\left(\dfrac{6}{7};-\dfrac{2}{7};-\dfrac{3}{7}\right)$.

Answer. $\vec{a}_0=\left(\dfrac{6}{7};-\dfrac{2}{7};-\dfrac{3}{7}\right)$.

3. Find the magnitude of the sum and difference of vectors $\vec{a}=(3;-5;8)$ and $\vec{b}=(-1;1;-4)$.

Solution. Let's find the sum of vectors $\vec{a}$ and $\vec{b}$:

$\vec{a}+\vec{b}=(3;-5;8)+(-1;1;-4)=(2;-4;4)$.

Then $|\vec{a}+\vec{b}|=\sqrt{2^2+(-4)^2+4^2}=\sqrt{4+16+16}=\sqrt{36}=6$.

The difference of vectors $\vec{a}$ and $\vec{b}$:

$\vec{a}-\vec{b}=(3;-5;8)-(-1;1;-4)=(4;-6;12)$, and

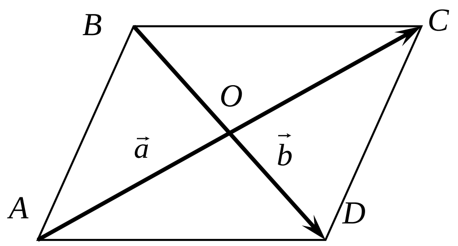
$|\vec{a}-\vec{b}|=\sqrt{4^2+(-6)^2+12^2}=\sqrt{16+36+144}=\sqrt{196}=14$.

Answer. $|\vec{a}+\vec{b}|=6$; $|\vec{a}-\vec{b}|=14$.

4. The vectors $\overrightarrow{AC}=\vec{a}$ and $\overrightarrow{BD}=\vec{b}$ are given in parallelogram *ABCD*, that coincide with its diagonal: Find the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DA}$, that coincide with the sides of parallelogram, if:

$$\vec{a}=(1;-2;4),\quad \vec{b}=(2;3;0).$$

Solution.



Let the diagonals intersect at the point O. Using the parallelogram rule of vectors' addition $\overrightarrow{AB}+\overrightarrow{BO}=\overrightarrow{AO}$, therefore $\overrightarrow{AB}=\overrightarrow{AO}-\overrightarrow{BO}$. Since

$\overrightarrow{BO}=\frac{1}{2}\overrightarrow{BD}=\frac{1}{2}\vec{b},\quad \overrightarrow{AO}=\frac{1}{2}\overrightarrow{AC}=\frac{1}{2}\vec{a}$, то

$\overrightarrow{AB}=\frac{1}{2}\vec{a}-\frac{1}{2}\vec{b}=\frac{1}{2}(\vec{a}-\vec{b})$.

In the components form $\overrightarrow{AB}=\frac{1}{2}\left(\left(1;-2;4\right)-\left(2;3;0\right)\right)=\frac{1}{2}\left(-1;-5;4\right)=\left(-\frac{1}{2};-\frac{5}{2};2\right)$.

Vector $\overrightarrow{BC}$ can be found using the subtraction rule of vectors: $\overrightarrow{BC}=\overrightarrow{AC}-\overrightarrow{AB}$, therefore, $\overrightarrow{BC}=\left(1;-2;4\right)-\left(-\frac{1}{2};-\frac{5}{2};2\right)=\left(\frac{3}{2};\frac{1}{2};2\right)$. Since vectors $\overrightarrow{CD}$ and $\overrightarrow{AB}$ are oppositely directed, then $\overrightarrow{CD}=-\overrightarrow{AB}=-\left(-\frac{1}{2};-\frac{5}{2};2\right)=\left(\frac{1}{2};\frac{5}{2};-2\right)$. Similarly:

$\overrightarrow{DA}=-\overrightarrow{BC}=-\left(\frac{3}{2};\frac{1}{2};2\right)=\left(-\frac{3}{2};-\frac{1}{2};-2\right)$.

Answer.

$\overrightarrow{AB}=\left(-\frac{1}{2};-\frac{5}{2};2\right),\overrightarrow{BC}=\left(\frac{3}{2};\frac{1}{2};2\right),\overrightarrow{CD}=\left(\frac{1}{2};\frac{5}{2};-2\right),\overrightarrow{DA}=\left(-\frac{3}{2};-\frac{1}{2};-2\right)$.

4. The scalar product of two vectors $\vec{a} \cdot \vec{b}$

Definition. The scalar product of two vectors $\vec{a}$ and $\vec{b}$ is a number, that is equal to the product of the magnitude of these two vectors on the cosine of angle between them, that is

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos \varphi, \qquad (4.1)$$

where φ is the angle between the vectors $\vec{a}$ and $\vec{b}$. The scalar product is also known as inner or dot product.

The properties of the scalar product:

1) $$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \qquad (4.2)$$

2) $$\left(k\vec{a}\right) \cdot \vec{b} = \vec{a} \cdot \left(k\vec{b}\right) = k\left(\vec{a} \cdot \vec{b}\right) \qquad (4.3)$$

3) $$\left(\vec{a} + \vec{b}\right) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} \qquad (4.4)$$

4) $$\vec{a} \cdot \vec{a} = |\vec{a}|^2 \qquad (4.5)$$

The component's form of the scalar product:

If vectors $\vec{a}$ and $\vec{b}$ are given in the component form, that is $\vec{a} = \left(a_x, a_y, a_z\right)$, $\vec{b} = \left(b_x, b_y, b_z\right)$, then their scalar product is calculated by the formula:

$$\vec{a} \cdot \vec{b} = a_X \cdot b_X + a_y \cdot b_y + a_Z \cdot b_Z \qquad (4.6)$$

The scalar product of two vectors is equal to the sum of product of the respective components.

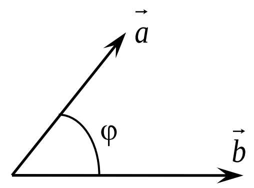


Figure 4.1

The applications of the scalar product:

1) **Finding the cosine of angle between the vectors.** Using the definition of scalar product (4.1) and the component form (4.6), one gets the formulas to find the cosine of angle between the vectors $\vec{a}$ and $\vec{b}$ (Fig. 1):

$$\cos \varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} \quad \text{or} \quad \cos \varphi = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2} \cdot \sqrt{b_x^2 + b_y^2 + b_z^2}} \qquad (4.7)$$

2) **Determination of perpendicularity** of vectors. Since $\cos \varphi = 0$ when $\varphi = 90°$, then by the formula (4.1) one gets the condition of perpendicularity (or orthogonality) of vectors.

Vectors $\vec{a}$ and $\vec{b}$ are perpendicular if and only if, when their scalar product is equal to zero:

$$\vec{a} \perp \vec{b} \quad \Leftrightarrow \quad \vec{a} \cdot \vec{b} = 0 \qquad (4.8)$$

If vectors $\vec{a}$ are $\vec{b}$ given by their components, that is $\vec{a}=\left(a_x,a_y,a_z\right)$, $\vec{b}=\left(b_x,b_y,b_z\right)$ then one gets **the condition of perpendicularity in the components form**:

$$\vec{a}\perp\vec{b} \quad \Leftrightarrow \quad a_xb_x+a_yb_y+a_zb_z=0 \tag{4.9}$$

Therefore, using the scalar product it is possible to determine the type of angle between the vectors:

- If $\vec{a}\cdot\vec{b}>0$, then $\varphi<90°$ (φ is acute angle);
- If $\vec{a}\cdot\vec{b}=0$, then $\varphi=90°$ (φ is right angle);
- If $\vec{a}\cdot\vec{b}<0$, then $90°<\varphi<180°$. φ is obtuse angle.

3) **Finding the projection** of vector $\vec{a}$ on the vector $\vec{b}$. Using the formulas (2.1) and the definition of scalar product, one gets the formula to find the projection of vector on the vector without the usage of angle between them:

$$np_{\vec{b}}\vec{a}=\frac{\vec{a}\cdot\vec{b}}{\left|\vec{b}\right|}. \tag{4.10}$$

4) **Finding the total cost of products**. Let a_x,a_y,a_z are the quantities of three products, and b_x,b_y,b_z are their prices, then $\vec{a}=\left(a_x,a_y,a_z\right)$ is the vector of the volume of the products, $\vec{b}=\left(b_x,b_y,b_z\right)$ is the price vector. The scalar product $\vec{a}\cdot\vec{b}$ determines the total cost of the products.

Questions for self-check

1) Give the definition of the scalar product of two vectors.
2) Will the result change, if interchange the vectors in the scalar product?
3) The product of scalar product of vector on itself is equal to...
4) How it is possible to find the scalar product in the component form?
5) If the scalar product of two vectors is equal to zero, then these vectors are…
6) If the scalar product of two vectors is positive, then the angle between these vectors …
7) If two vectors form the obtuse angle, then the scalar product of these vectors is …
8) If two vectors are perpendicular, then their scalar product is equal to…
9) Write down the condition of perpendicularity of vectors in the component form.
10) Write down the formula of finding the cosine of angle between the vectors.
11) Which formula can be used to find the projection of vector $\vec{a}$ on vector $\vec{b}$?

Solved problems

1. The angle between the vectors $\vec{a}$ and $\vec{b}$ is $\varphi = \frac{2\pi}{3}$, the lengths of vectors $|\vec{a}| = 4$, $|\vec{b}| = 3$. Calculate: 1) $\vec{a} \cdot \vec{b}$; 2) $\vec{a}^{\,2}$.

Solution. 1) Using the definition of scalar product (4.1):

$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi = 4 \cdot 3 \cdot \cos\frac{2\pi}{3} = 12 \cdot \left(-\frac{1}{2}\right) = -6.$

2) By formula (4.5) $\vec{a}^{\,2} = |\vec{a}|^2 = 16.$

Answer. 1) $\vec{a} \cdot \vec{b} = -6$, 2) $\vec{a}^{\,2} = 16.$

2. Given vectors $\vec{c} = (2;-1;-2)$, $\vec{d} = (12;-6;4)$. Calculate $\vec{c} \cdot \vec{d}$.

Solution. Since vectors are given in the component form, then let's apply the formula (4.6): $\vec{c} \cdot \vec{d} = 2 \cdot 12 + (-1) \cdot (-6) + (-2) \cdot 4 = 24 + 6 - 8 = 22.$

Answer. 22.

3. Calculate the cosine of angle, formed by the vectors $\vec{a} = (2;-4;4)$ and $\vec{b} = (-3;2;6)$.

Solution. By the formula (4.7) $\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$.

Let's find $\vec{a} \cdot \vec{b} = 2 \cdot (-3) + (-4) \cdot 2 + 4 \cdot 6 = -6 - 8 + 24 = 10;$

$|\vec{a}| = \sqrt{2^2 + (-4)^2 + 4^2} = \sqrt{4 + 16 + 16} = \sqrt{36} = 6,$

$|\vec{b}| = \sqrt{(-3)^2 + 2^2 + 6^2} = \sqrt{9 + 4 + 36} = \sqrt{49} = 7$. Then $\cos\varphi = \frac{10}{6 \cdot 7} = \frac{10}{42} = \frac{5}{21}$.

Answer. $\cos\varphi = \frac{5}{21}$.

4. Given the vertices of triangle $A(0;-1)$, $B(-3;-1)$ and $C(4;1)$. Determine theit internal angle at the vertex *B*.

Solution. Since the angle *B* is formed by the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$ (Fig. 2), then $\cos(\angle B) = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}| \cdot |\overrightarrow{BC}|}$. Let's find $\overrightarrow{BA} = (0 - (-3); -1 - (-1)) = (-3;0)$,

$\overrightarrow{BC} = (4 - (-3); 1 - (-1)) = (7;2)$. Then $\overrightarrow{BA} \cdot \overrightarrow{BC} = 3 \cdot 7 + 0 \cdot 2 = 21$.

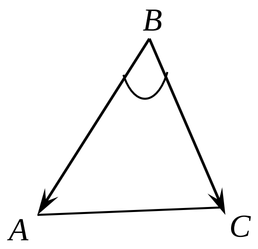


Figure 4. 2

Let's find the magnitudes of vectors:

$$\left|\overrightarrow{BA}\right| = \sqrt{(-3)^2 + 0^2} = 3, \quad \left|\overrightarrow{BC}\right| = \sqrt{7^2 + 2^2} = \sqrt{49+4} = \sqrt{53}.$$

Then $\cos(\angle B) = \dfrac{21}{3\cdot\sqrt{53}} = \dfrac{7}{\sqrt{53}} = \dfrac{7\sqrt{53}}{53}$.

Therefore, $\angle B = \arccos\left(\dfrac{7\sqrt{53}}{53}\right)$.

The value $\arccos\dfrac{7\sqrt{53}}{53}$ can by found, using the calculator and Bradis tables: $\dfrac{7\sqrt{53}}{53} \approx 0{,}9615$; $\arccos\dfrac{7\sqrt{53}}{53} \approx 16°$.

Answer. $\angle B \approx 16°$.

5. Find the projection of vector $\vec{a}$ on the axis of vector $\vec{b}$, if $\vec{a} = (5;2;5)$, $\vec{b} = (2;-1;2)$.

Solution. By formula (4.10): $np_{\vec{b}}\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{\left|\vec{b}\right|}$.

Let's find: $\vec{a}\cdot\vec{b} = 5\cdot2 + 2\cdot(-1) + 5\cdot2 = 10 - 2 + 10 = 18$,

$\left|\vec{b}\right| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{4+1+4} = \sqrt{9} = 3.$ Then $np_{\vec{b}}\vec{a} = \dfrac{18}{3} = 6$.

Answer. 6.

6. Given the vertices of quadrilateral $A(2;-1;2)$, $B(2;5;0)$, $C(-3;2;1)$ and $D(-4;-4;3)$. Prove, that its diagonals AC and BD are mutually perpendicular.

Solution. Let's find the components of vectors $\overrightarrow{AC}$ and $\overrightarrow{BD}$.

$\overrightarrow{AC} = (-3-2; 2-(-1); 1-2) = (-5;3;-1)$, $\overrightarrow{BD} = (-4-2; -4-5; 3-0) = (-6;-9;3)$.

Let's find the scalar product of these vectors:

$\overrightarrow{AC}\cdot\overrightarrow{BD} = (-5)\cdot(-6) + 3\cdot(-9) + (-1)\cdot3 = 30 - 27 - 3 = 0.$

Since $\overrightarrow{AC}\cdot\overrightarrow{BD} = 0$, then vectors $\overrightarrow{AC}$ and $\overrightarrow{BD}$ intersect at right angle.

Answer. The diagonals AC and BD are mutually perpendicular.

5. The vector product of two vectors $\vec{a} \times \vec{b}$

Definition. The triple of non-coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is called the **right-handed (left-handed)**, if the nearest rotation from $\vec{a}$ to $\vec{b}$, that is observed from the terminal point of vector $\vec{c}$, is performed counter clockwise (clockwise).

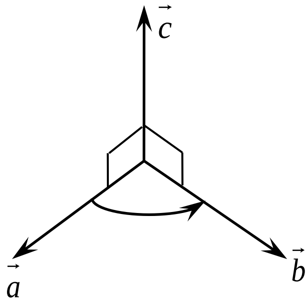


Figure 5.1

Definition. The vector product of vector $\vec{a}$ on the vector $\vec{b}$ is called such vector $\vec{c} = \vec{a} \times \vec{b}$, which length and direction are defined by the following conditions:

1) $\left|\vec{c}\right| = \left|\vec{a}\right| \cdot \left|\vec{b}\right| \cdot \sin\varphi$; (5.1)

2) $\vec{c} \perp \vec{a}$, $\vec{c} \perp \vec{b}$, moreover, vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ in this order form the right-handed triple (Fig. 1).

The properties of vector product:

1. $\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$. (5.2)

2. $k\left(\vec{a} \times \vec{b}\right) = \left(k\vec{a}\right) \times \vec{b} = \vec{a} \times \left(k\vec{b}\right)$. (5.3)

3. $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$. (5.4)

4. $\vec{a} \times \vec{a} = 0$. (5.5)

The component form of vector product

If vectors $\vec{a}$ and $\vec{b}$ are given in the component form, that is, $\vec{a} = \left(a_x, a_y, a_z\right)$, $\vec{b} = \left(b_x, b_y, b_z\right)$, then their vector product is equal to the determinant, which first row is formed by the by the unit vectors of the Cartesian rectangular coordinate system, and the second and the third row are formed by the components of vectors, that are being multiplied:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}. \qquad (5.6)$$

The applications of the vector product

1) The determination of the collinearity of vectors.

Since $\sin\varphi = 0$, if $\varphi = 0$ or $\varphi = \pi$, then from (5.1) one gets the **condition for collinearity of vectors**:

vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if, when their vector product is equal to zero:

$$\vec{a} \parallel \vec{b} \quad \Leftrightarrow \quad \vec{a} \times \vec{b} = 0 \tag{5.7}$$

If vectors $\vec{a}$ and $\vec{b}$ are given by their components, that is $\vec{a} = \left(a_x, a_y, a_z\right)$, $\vec{b} = \left(b_x, b_y, b_z\right)$, then, using the properties of determinants, one gets the **condition of collinearity in the component form**:

$$\vec{a} \parallel \vec{b} \quad \Leftrightarrow \quad \frac{a_x}{b_x} = \frac{a_y}{b_y} = \frac{a_z}{b_z}. \tag{5.8}$$

If vectors are collinear, then their respective components are proportional.

2) To find the area of parallelogram (Fig. 2) or triangle (Fig. 3), built on the vectors $\vec{a}$ and $\vec{b}$:

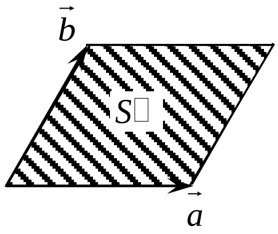


Figure 5.2

$$S_{\square} = \left|\vec{a} \times \vec{b}\right| \tag{5.9}$$

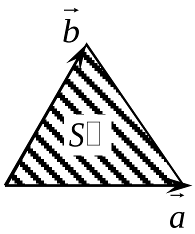


Figure 5.3

$$S_{\triangle} = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| \tag{5.10}$$

Questions for self-check

1) Give the definition of the right-handed triple of vectors.
2) Give the definition of the vector product of two vectors.
3) How is the vector product of two vectors denoted?
4) Write down the formula, which is used to find the length of vector $\vec{c} = \vec{a} \times \vec{b}$.
5) Will the result change, if one interchanges the vectors in the vector product?
6) The vector product of vector on itself is equal to ...
7) How can we find the vector product, if vectors are given by their components?
8) If vector product of two vectors is equal to zero, then these vectors …
9) If vectors are collinear, then their components …
10) What is the geometrical application of the vector product of two vectors?

Solved problems

1. Check whether the points *A*, *B*, *C* and *D* are the vertices of the trapezium *ABCD*, if:

$A(3;0;4)$, $B(2;-1;1)$, $C(3;0;-3)$, $D(7;4;-12)$.

Solution. If *ABCD* is the trapezium, then their two sides are parallel, and another two are not. Then let's show that, for instance, the vectors $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are collinear, and $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are not collinear. Let's find the components of vectors $\overrightarrow{BC}$ and $\overrightarrow{AD}$.

$\overrightarrow{BC} = (3-2;\, 0-(-1);\, -3-1) = (1;1;-4)$, $\overrightarrow{AD} = (7-3;\, 4-0;\, -12-4) = (4;4;-16)$.

Let's write down a proportion from the components of these vectors: $\frac{1}{4} = \frac{1}{4} = \frac{-4}{-16}$; $\lambda = \frac{1}{4}$. Therefore, vectors $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are collinear. Let's find the components of vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

$\overrightarrow{AB} = (2-3;\, -1-0;\, 1-4) = (-1;-1;-3)$, $\overrightarrow{CD} = (7-3;\, 4-0;\, -12-(-3)) = (4;4;-9)$.

Since $\frac{-1}{4} = \frac{-1}{4} \neq \frac{-3}{-9}$, then $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are not collinear.

Therefore, the points *A*, *B*, *C* and *D* are the vertices of the trapezium *ABCD*.

Answer. Yes.

2. Find the magnitude of vector product $\left|\vec{a} \times \vec{b}\right|$, if $\left|\vec{a}\right| = 3$, $\left|\vec{b}\right| = 10$, $\left(\widehat{\vec{a},\vec{b}}\right) = \frac{\pi}{6}$.

Solution. By the formula (5.1):

$\left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right| \cdot \left|\vec{b}\right| \cdot \sin\left(\widehat{\vec{a},\vec{b}}\right)$. Therefore, $\left|\vec{a} \times \vec{b}\right| = 3 \cdot 10 \cdot \sin\frac{\pi}{6} = 30 \cdot \frac{1}{2} = 15$.

Answer. 15.

3. Find the vector product $\vec{a} \times \vec{b}$, if:

1) $\vec{a} = (1;2;-2)$, $\vec{b} = (8;6;4)$;

2) $\vec{a} = 7\vec{i} + 4\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} - 3\vec{k}$.

Solution.

1) The vectors are given in the component form, thus, let's use (5.6):

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -2 \\ 8 & 6 & 4 \end{vmatrix} = 8\vec{i} + 6\vec{k} - 16\vec{j} - 16\vec{k} + 12\vec{i} - 4\vec{j} = 20\vec{i} - 20\vec{j} - 10\vec{k} \cdot$$

Therefore, $\vec{a} \times \vec{b} = (20;-20;-10)$.

2) Similarly, as in the clause 1), one gets:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 4 & 1 \\ 2 & 0 & -3 \end{vmatrix} = -12\vec{i} + 0\vec{k} + 2\vec{j} - 8\vec{k} + 0\vec{i} + 21\vec{j} = -12\vec{i} + 23\vec{j} - 8\vec{k}.$$

Therefore, $\vec{a}\times\vec{b}=(-12;23;-8)$.

Answer. 1) $\vec{a}\times\vec{b}=(20;-20;-10)$; 2) $\vec{a}\times\vec{b}=(-12;23;-8)$.

4. Given: $|\vec{a}|=10$, $|\vec{b}|=2$, $\vec{a}\cdot\vec{b}=12$. Calculate $|\vec{a}\times\vec{b}|$.

Solution. Since the scalar product is known, then let's find $\cos\left(\widehat{\vec{a},\vec{b}}\right)$:

$\cos\left(\widehat{\vec{a},\vec{b}}\right)=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}=\dfrac{12}{10\cdot 2}=\dfrac{12}{20}=\dfrac{3}{5}$. Then from the formula, known from the trigonometry, $\sin\left(\widehat{\vec{a},\vec{b}}\right)=\pm\sqrt{1-\cos^2\left(\widehat{\vec{a},\vec{b}}\right)}=\pm\sqrt{1-\left(\dfrac{3}{5}\right)^2}=\pm\sqrt{1-\dfrac{9}{25}}=\pm\sqrt{\dfrac{16}{25}}=\pm\dfrac{4}{5}$.

Since the angle φ between the vectors is in the range from 0 to π, then $\sin\left(\widehat{\vec{a},\vec{b}}\right)\geq 0$, therefore, $\sin\left(\widehat{\vec{a},\vec{b}}\right)=\dfrac{4}{5}$.

Hence, from the formula (5.1) $|\vec{a}\times\vec{b}|=|\vec{a}|\cdot|\vec{b}|\cdot\sin\left(\widehat{\vec{a},\vec{b}}\right)=10\cdot 2\cdot\dfrac{4}{5}=16.$

Answer. 16.

5. Determine, for which λ the vectors $\vec{a}=(1;\lambda)$ and $\vec{b}=(4;-16)$ are collinear.

Solution. If vectors $\vec{a}$ and $\vec{b}$ are collinear, then their components are proportional. Let's write down the proportion: $\dfrac{1}{4}=\dfrac{\lambda}{-16}$. Hence, $\lambda=-4$.

Answer. −4.

6. For which values of α and β the point $C(\alpha;\beta;3)$ lie on the line AB, if $A(0;0;1)$, $B(-6;8;5)$?

Solution. If point C lie on the line AB, then vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear. Let's find the components of these vectors:
$\overrightarrow{AB}=(-6-0;8-0;5-1)=(-6;8;4)$, $\overrightarrow{AC}=(\alpha-0;\beta-0;3-1)=(\alpha;\beta;2)$. Let's write down the proportion: $\dfrac{\alpha}{-6}=\dfrac{\beta}{8}=\dfrac{2}{4}$. Hence, $\alpha=-3, \beta=4$.

Answer. $\alpha=-3, \beta=4$.

7. Calculate the area of triangle ABC, if $A(4;2;3)$, $B(5;1;2)$, $C(6;5;8)$.

Solution. By the formula (5.10) $S_{\Delta ABC}=\dfrac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$.

Let's find at first $\overrightarrow{AB}\times\overrightarrow{AC}$. $\overrightarrow{AB}=(5-4;1-2;2-3)=(1;-1;-1)$,
$\overrightarrow{AC}=(6-4;5-2;8-3)=(2;3;5)$.

Then $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & -1 \\ 2 & 3 & 5 \end{vmatrix} = -5\vec{i} + 3\vec{k} - 2\vec{j} + 2\vec{k} + 3\vec{i} - 5\vec{j} = -2\vec{i} - 7\vec{j} + 5\vec{k}$.

Hence, $S_{\Delta ABC} = \frac{1}{2}\sqrt{(-2)^2 + (-7)^2 + 5^2} = \frac{1}{2}\sqrt{4 + 49 + 25} = \frac{1}{2}\sqrt{78}.$

Answer. $\frac{\sqrt{78}}{2}$ (square units).

6. The scalar triple product of three vectors $\vec{a}\vec{b}\vec{c}$

Definition. The scalar triple product (or mixed product) of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is called a number, which is equal to the scalar product of the vector product of vectors $\vec{a}$ and $\vec{b}$ on the vector $\vec{c}$:

$$\vec{a}\vec{b}\vec{c} = \left(\vec{a} \times \vec{b}\right) \cdot \vec{c} \qquad (6.1)$$

The properties of scalar triple product $\vec{a}\vec{b}\vec{c}$

1) $\vec{a}\vec{b}\vec{c} = \vec{b}\vec{c}\vec{a} = \vec{c}\vec{a}\vec{b} = -\vec{c}\vec{b}\vec{a} = -\vec{b}\vec{a}\vec{c} = -\vec{a}\vec{c}\vec{b}.$ (6.2)
2) If $\vec{a}\vec{b}\vec{c} > 0$, then $\vec{a}$, $\vec{b}$ and $\vec{c}$ form the right-handed triple.
3) If $\vec{a}\vec{b}\vec{c} < 0$, then $\vec{a}$, $\vec{b}$ and $\vec{c}$ form the left-handed triple.

The component form of scalar triple product

If vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are given by their components, that is $\vec{a} = \left(a_x, a_y, a_z\right)$, $\vec{b} = \left(b_x, b_y, b_z\right)$, $\vec{c} = \left(c_x, c_y, c_z\right)$, then their mixed product is equal to the determinant, formed by the components of these vectors:

$$\vec{a}\,\vec{b}\,\vec{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \qquad (6.3)$$

The applications of scalar triple product

1) To determine the coplanarity of vectors.

Vectors $\vec{a}$***,*** $\vec{b}$, $\vec{c}$ ***are coplanar if and only if, when their*** scalar triple product is equal to zero***:***

$$\vec{a},\ \vec{b},\ \vec{c} \text{ are coplanar} \quad \Leftrightarrow \quad \vec{a}\,\vec{b}\,\vec{c} = 0 \qquad (6.4)$$

If vectors $\vec{a}$ and $\vec{b}$ are given by their components, that is $\vec{a}=\left(a_x,a_y,a_z\right)$, $\vec{b}=\left(b_x,b_y,b_z\right)$, then one gets **the condition of coplanarity in the component form**:

$$\vec{a},\ \vec{b},\ \vec{c}\ \text{ are coplanar} \quad \Leftrightarrow \quad \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}=0 \qquad (6.5)$$

If vectors are coplanar, then the determinant, formed from their components, is equal to zero.

2) To find the volume of the **parallelepiped**, built on the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ (Fig. 1):

$$V_{par}=\left|\vec{a}\vec{b}\vec{c}\right|. \qquad (6.6)$$

3) To find **the volume of the pyramid**, built on the on the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ (Fig. 2):

$$V_{pyr}=\frac{1}{6}\left|\vec{a}\vec{b}\vec{c}\right|. \qquad (6.7)$$

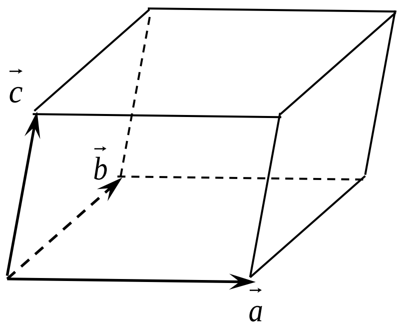


Figure 6.1

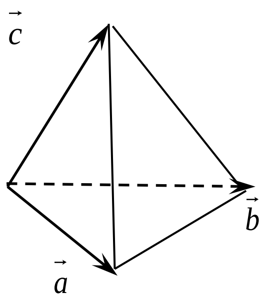


Figure 6.2

Questions for self-check

1) Give the definition of the scalar triple product of three vectors.
2) If the scalar triple product of three vectors is positive, then these vectors form…
3) By which formula can the scalar triple product of three vectors be found, if their components are given?
4) If three vector are coplanar, then their scalar triple product is equal to …
5) What is the geometrical application of scalar triple product?

Solved problems

1. Determine, for whcih λ the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, if:

$$\vec{a}=\left(1;\lambda;3\right),\quad \vec{b}=\left(4;5;\lambda\right),\quad \vec{c}=\left(\lambda;1;-4\right).$$

Solution. The vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, if their triple scalar product of these vectors is equal to zero. Let's use the component form of the triple scalar product:

$$\vec{a}\vec{b}\vec{c} = \begin{vmatrix} 1 & \lambda & 3 \\ 4 & 5 & \lambda \\ \lambda & 1 & -4 \end{vmatrix} = -20 + 12 + \lambda^3 - 15\lambda - \lambda + 16\lambda = \lambda^3 - 8.$$

From the condition of coplanarity $\lambda^3 - 8 = 0$, one gets $\lambda = 2$.

Answer. 2.

2. Determine the volume of parallelepiped *ABCD*, if:

$$A(0;2;-1),\quad B(-3;4;-1),\quad C(2;1;-2),\quad D(1;10;1).$$

Solution. Since the parallelepiped *ABCD* is build on the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$, then its volume is equal to the absolute value of scalar triple product of these vectors: $V = \left|\overrightarrow{AB} \cdot \overrightarrow{AC} \cdot \overrightarrow{AD}\right|$.

The components of vectors: $\overrightarrow{AB} = (-3;2;0)$, $\overrightarrow{AC} = (2;-1;-1)$, $\overrightarrow{AD} = (1;8;2)$.

$$\overrightarrow{AB} \cdot \overrightarrow{AC} \cdot \overrightarrow{AD} = \begin{vmatrix} -3 & 2 & 0 \\ 2 & -1 & -1 \\ 1 & 8 & 2 \end{vmatrix} = 6 + 0 - 2 - 0 - 24 - 8 = -28.$$

Hence, by formula (6.6) $V = |-28| = 28$.

Answer. 28 (cubic units).

7. The concept of *n*-dimensional vector space

Definition. The ordered sequence of n numbers $(a_1, a_2, ..., a_n)$ is called ***n*-dimensional vector**.

Definition. The vector $\vec{0} = (0, 0, ..., 0)$ is called the **zero**, or **null-vector of** *n*-dimensional vector space.

Definition. The vectors of *n*-dimensional space $\vec{a} = (a_1, a_2, ..., a_n)$ and $\vec{b} = (b_1, b_2, ..., b_n)$ are called **equal**, if all their respective components are equal:

$$a_i = b_i,\ \ i = \overline{1,n}. \tag{7.1}$$

Let's introduce the linear operations on *n*-dimensional vectors:

1) The product of vector $\vec{a} = (a_1, a_2, ..., a_n)$ on a number k is called the vector

$$k\vec{a} = (ka_1, ka_2, ..., ka_n). \tag{7.2}$$

2) The sum of vectors $\vec{a} = (a_1, a_2, ..., a_n)$ and $\vec{b} = (b_1, b_2, ..., b_n)$ is called the vector

$$\vec{a} + \vec{b} = (a_1 + b_1,\ a_2 + b_2,\ ...,\ a_n + b_n). \tag{7.3}$$

3) The difference of vectors $\vec{a}=(a_1, a_2, ..., a_n)$ and $\vec{b}=(b_1, b_2, ..., b_n)$ is called the vector

$$\vec{a}-\vec{b}=(a_1-b_1,\ a_2-b_2,\ ...,\ a_n-b_n). \qquad (7.4)$$

Definition. The set of all *n*-dimensional vectors, for which the linear operations are introduced, are called ***n*-dimensional linear vector space**.

The length (norm) of *n*-dimensional vector $\vec{a}=(a_1, a_2, ..., a_n)$ can be found by the formula

$$|\vec{a}|=\sqrt{a_1^{\ 2}+a_2^{\ 2}+...+a_n^{\ 2}}\ . \qquad (7.5)$$

The scalar product of two *n*-dimensional vectors $\vec{a}=(a_1, a_2, ..., a_n)$ and $\vec{b}=(b_1, b_2, ..., b_n)$ can be found by the formula

$$\vec{a}\cdot\vec{b}=a_1\cdot b_1+a_2\cdot b_2+...+a_n\cdot b_n. \qquad (7.6)$$

Questions for self-check

1) Give the definition of *n*-dimensional vector.
2) Write down the *n*-dimensional zero vector.
3) The vectors of *n*-dimensional space are called equal, if…
4) Write down the formula of multiplication of *n*-dimensional vector on a number.
5) Write down the formula of addition of two *n*-dimensional vectors.
6) Provide a definition of *n*-dimensional linear vector space.
7) The length of *n*-dimensional vector can be calculated by the formula…
8) The scalar product of two *n*-dimensional vectors can be found by the formula…

Solved problems

1. Given vectors: $\vec{a}=(1;-3;4;0)$, $\vec{b}=(5;4;2;-7)$. Find:

a) $\vec{a}+\vec{b}$; b) $4\vec{a}-2\vec{b}$; c) $\vec{a}\cdot\vec{b}$.

Solution

a)

$\vec{a}+\vec{b}=(1;-3;4;0)+(5;4;2;-7)=(6;1;6;-7)$.

b)

$4\vec{a}-2\vec{b}=4\cdot(1;-3;4;0)-2\cdot(5;4;2;-7)=(4;-12;16;0)-(10;8;4;-14)=(-6;-20;12;14)$

c)

$\vec{a}\cdot\vec{b}=(1;-3;4;0)\cdot(5;4;2;-7)=1\cdot5+(-3)\cdot4+4\cdot2+0\cdot(-7)=5-12+8+0=1$.

Answer. a) $(6;1;6;-7)$; b) $(-6;-20;12;14)$; c) 1.

2. Find the norm of vector $\overrightarrow{AB}$, if $A(-3;2;1;0;5)$ and $B(7;6;1;0;0)$.

Solution. Find the components of vector $\overrightarrow{AB}$:

$\overrightarrow{AB}=(7-(-3);6-2;1-1;0-0;0-5)=(10;4;0;0;-5)$.

Then $\left|\overrightarrow{AB}\right| = \sqrt{10^2 + 4^2 + 0^2 + 0^2 + (-5)^2} = \sqrt{100+16+25} = \sqrt{141}$.

Answer. $\sqrt{141}$.

8. The linear independency of the system of vectors. The basis of vectors

Let $\overrightarrow{a_1}, \overrightarrow{a_2}, ..., \overrightarrow{a_n}$ are non-zero *n*-dimensional vectors.

Definition. The expression $k_1\overrightarrow{a_1} + k_2\overrightarrow{a_2} + ... + k_n\overrightarrow{a_n}$, where $k_1, k_2, \ldots, k_n$ are real number, is called **the linear combination** of vectors $\overrightarrow{a_1}, \overrightarrow{a_2}, ..., \overrightarrow{a_n}$.

Definition. If $k_1\overrightarrow{a_1} + k_2\overrightarrow{a_2} + ... + k_n\overrightarrow{a_n} = 0$ only when $k_1 = k_2 = \ldots = k_n = 0$, then the vectors $\overrightarrow{a_1}, \overrightarrow{a_2}, ..., \overrightarrow{a_n}$ are called **linear independent**.

Definition. If there exist such numbers $k_1, k_2, \ldots, k_n$, not all equal to zero, under which $k_1\overrightarrow{a_1} + k_2\overrightarrow{a_2} + ... + k_n\overrightarrow{a_n} = 0$, then the vectors $\overrightarrow{a_1}, \overrightarrow{a_2}, ..., \overrightarrow{a_n}$ are called **linear dependent**.

If vectors $\overrightarrow{a_1}, \overrightarrow{a_2}, ..., \overrightarrow{a_n}$ are linear dependent, then at least one of them can be expressed as a linear combination of all others, for instance,

$$\overrightarrow{a_n} = \lambda_1\overrightarrow{a_1} + \lambda_2\overrightarrow{a_2} + ... + \lambda_{n-1}\overrightarrow{a_{n-1}} \tag{8.1}$$

where $\lambda_i = -\dfrac{k_i}{k_n}, \quad i = \overline{1, n-1}$.

Definition. The maximum number of linear independent vectors of the space is called the **dimension of space**. *n*-dimensional space is denoted as $\mathbb{R}^n$.

Definition. The ordered sequence of *n* linear independent vectors of the space $\mathbb{R}^n$ form the **basis** of this space.

Therefore, from the formulated definitions one can conclude that:

- the basis on the line *OX* (at $\mathbb{R}^1$) is any non-zero vector;
- the basis on the plane *OXY* (at $\mathbb{R}^2$) are any two non-colinear vectors;
- the basis on the space *OXYZ* (at $\mathbb{R}^3$) are any three non-complanar vectors.

Definition. If the basis vectors are mutually perpendicular, then such a basis is called **orthogonal**.

Definition. If the basis vectors are unit and mutually perpendicular, then such a basis is called **orthonormal**.

.

The unit vectors $\vec{i}, \vec{j}, \vec{k}$, connected with the axes *OX*, *OY* and *OZ*, are the orthonormal basis of the Cartesian rectangular coordinate system *OXYZ*. If a_x, a_y, a_z

are the components of vector $\vec{a}$ in the bais $\vec{i}, \vec{j}, \vec{k}$, then the vector $\vec{a}$ can be expressed as the linear combination

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}. \qquad (8.2)$$

Definition. If the angles between the basis vectors are arbitrary, then the basis is called **arbitrary** and is denoted as $\vec{e_1}, \vec{e_2}, ..., \vec{e_n}$.

If vector $\vec{a}$ is expressed in the form of linear combination of arbitrary basis vectors $\vec{e_1}, \vec{e_2}, ..., \vec{e_n}$, then one says that, it is decomposition of vector over the basis $\vec{e_1}, \vec{e_2}, ..., \vec{e_n}$.

In the space $\mathbb{R}^3$ the decomposition of vector $\vec{a}$ in the arbitrary basis has the form:

$$\vec{a} = k_1\vec{e_1} + k_2\vec{e_2} + k_3\vec{e_3}. \qquad (8.3)$$

The numbers k_1, k_2, k_3 are called the components of vector $\vec{a}$ in the basis $\vec{e_1}, \vec{e_2}, \vec{e_3}$ or the coefficients of the decomposition.

Questions for self-check

1) What is called the linear combination of *n*-dimensional vectors?
2) Formulate the condition of linear independency of *n*-dimensional vectors.
3) Formulate the condition of linear dependency of *n*- dimensional vectors.
4) Provide the definition of the basis over $\mathbb{R}^n$.
5) If the basis vectors are mutually independent, then such a basis is called...
6) Which vectors do form the **orthonormal basis**?
7) How is the **orthonormal basis** of Cartesian system denoted?

Solved problems

1. Given four vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ over some basis. Show that the vectors $\vec{a}, \vec{b}, \vec{c}$ form the basis and find the components of vector $\vec{d}$ over this basis, if:

$$\vec{a} = (3;1;-1), \quad \vec{b} = (2;2;1), \quad \vec{c} = (1;-2;0), \quad \vec{d} = (4;-2;1).$$

Solution. Three vectors $\vec{a}, \vec{b}, \vec{c}$ form the basis over the space, if they are non-coplanar. Let's calculate the determinant of matrix, formed from the components of these vectors:

$$\Delta = \begin{vmatrix} 3 & 2 & 1 \\ 1 & 2 & -2 \\ -1 & 1 & 0 \end{vmatrix} = 3\cdot 2\cdot 0 + 1\cdot 1\cdot 1 + 2\cdot(-2)\cdot(-1) - (-1)\cdot 2\cdot 1 - 3\cdot 1\cdot(-2) - 1\cdot 2\cdot 0 =$$

$$= 1 + 4 + 2 + 6 = 13 \neq 0.$$

Since $\Delta \neq 0$, then vectors $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar and form the basis.

The decomposition of vector $\vec{d}$ over this basis can be represented in the form: $\vec{d} = \alpha\cdot\vec{a} + \beta\cdot\vec{b} + \gamma\cdot\vec{c}$, where α, β, γ are the sought numbers. Let's write down this equality in the component form:

(4; −2;1)= α ·(3;1;−1)+ β ·(2;2;1)+ γ ·(1;−2;0), or, using the (3.2), (3.3) and (3.1)

$$\begin{cases} 3\alpha + 2\beta + \gamma = 4; \\ \alpha + 2\beta - 2\gamma = -2; \\ -\alpha + \beta = 1. \end{cases}$$

One gets the system of three linear equations with three unknowns. Let's solve this system using Cramer's method.

$\Delta = 13;$

$$\Delta_1 = \begin{vmatrix} 4 & 2 & 1 \\ -2 & 2 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 4\cdot 2\cdot 0 + (-2)\cdot 1\cdot 1 + 2\cdot(-2)\cdot 1 - 1\cdot 2\cdot 1 - 1\cdot(-2)\cdot 4 - (-2)\cdot 2\cdot 0 =$$

$= -2 - 4 - 2 + 8 = 0;$

$$\Delta_2 = \begin{vmatrix} 3 & 4 & 1 \\ 1 & -2 & -2 \\ -1 & 1 & 0 \end{vmatrix} = 3\cdot(-2)\cdot 0 + 1\cdot 1\cdot 1 + 4\cdot(-2)\cdot(-1) - (-1)\cdot(-2)\cdot 1 - 1\cdot(-2)\cdot 3 - 1\cdot 4\cdot 0 =$$

$= 1 + 8 - 2 + 6 = 13;$

$$\Delta_3 = \begin{vmatrix} 3 & 2 & 4 \\ 1 & 2 & -2 \\ -1 & 1 & 1 \end{vmatrix} = 3\cdot 2\cdot 1 + 1\cdot 1\cdot 4 + 2\cdot(-2)\cdot(-1) - (-1)\cdot 2\cdot 4 - 1\cdot(-2)\cdot 3 - 1\cdot 2\cdot 1 =$$

$= 6 + 4 + 4 + 8 + 6 - 2 = 26.$

Hence, one gets $\alpha = \frac{\Delta_1}{\Delta} = \frac{0}{13} = 0,\ \beta = \frac{\Delta_2}{\Delta} = \frac{13}{13} = 1,\ \gamma = \frac{\Delta_3}{\Delta} = \frac{26}{13} = 2.$

Therefore, $\vec{d} = 0\cdot\vec{a} + 1\cdot\vec{b} + 2\cdot\vec{c} = \vec{b} + 2\cdot\vec{c}$, or $\vec{d} = \vec{b} + 2\vec{c}.$

Solution. $\vec{d} = \vec{b} + 2\vec{c}.$

9. Eigenvalues and eigenvectors of the matrix

Let $A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}$ is the square matrix of the nth order, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is the n-dimensional vector.

Definition. The non-zero vector X is called the **eigenvector** of the matrix A, if there exits such a number λ, that the product of matrix A on the vector X is equal to the product of the number λ on this vector:

$$AX = \lambda X. \tag{9.1}$$

The number λ is called its **eigenvalue** (or eigennumber) of the matrix A.

The matrix equation (9.1) can be written in the form

$$\left(A - \lambda E\right) X = 0. \tag{9.2}$$

Hence, $A - \lambda E = 0$. In the expanded form one gets the homogenous system of linear equations

$$\begin{cases} \left(a_{11} - \lambda\right) x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = 0, \\ a_{21} x_1 + \left(a_{22} - \lambda\right) x_2 + \ldots + a_{2n} x_n = 0, \\ \qquad \ldots \\ a_{n1} x_1 + a_{n2} x_2 + \ldots + \left(a_{nn} - \lambda\right) x_n = 0. \end{cases} \tag{9.3}$$

The system (9.3) has the non-zero solution, if its determinant is equal to zero. Therefore, to find the eigenvalues the **characteristic equation** can be used

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - \lambda \end{vmatrix} = 0. \tag{9.4}$$

The eigenvectors, which correspond to the found eigennumbers, are determined from the equation (9.2).

Definition. The square matrix, all eigenvectors of which are positive, is called the **positive defined** matrix.

Theorem. The square matrix $A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}$ is positively defined if and only if, when each of its diagonal minors is positive:

$$a_{11} > 0; \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0; \quad \ldots; \quad \begin{vmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{n1} & \ldots & a_{nn} \end{vmatrix} > 0.$$

Questions for self-check

1) Give the definition of the eigenvector of the square matrix.

2) Write down the matrix equation, that expresses the definition of eigenvector of the square matrix.
3) Write down the characteristic equation that allows determining the eigennumbers of the square matrix.
4) If all eigennumbers of matrix are positive numbers, then such a matrix is called …
5) Check whether the matrix $A=\begin{pmatrix} 1 & -3 \\ 5 & -1 \end{pmatrix}$ is positively defined?

Solved problems

1. Check, whether the matrix $A=\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ is positively defined.

Solution. One can check whether the matrix is positively defined by calculating the minors:

$$a_{11}=1>0; \qquad \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix}=6>0.$$

Therefore, the matrix A is positively defined.

2. Find the eigennumbers and eigenvectors of the matrix $\begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$.

Solution. Let's write down the characteristic equation $\begin{vmatrix} 3-\lambda & 4 \\ 5 & 2-\lambda \end{vmatrix}=0$.

Therefore, $(3-\lambda)\cdot(2-\lambda)-20=0$. Hence, $6-3\lambda-2\lambda+\lambda^2-20=0$. Thus, $\lambda^2-5\lambda-14=0$. Then $\lambda_1=-2$, $\lambda_2=7$. Let's find the eigenvectors that correspond to these eigenvalues.

Let's $\lambda_1=-2$, and the sought vector $X=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. From the equation (9.2) one gets $(A+2E)\cdot X=0$, or $\left(\begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}+\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right)\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=0$, $\begin{pmatrix} 5 & 4 \\ 5 & 4 \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=0$, hence, one gets the only one equation $5x_1+4x_2=0$. Let's express x_1 through x_2: $5x_1=-4x_2$, $x_1=\frac{-4}{5}x_2$. If $x_2=c$, then $X=\begin{pmatrix} -\frac{4}{5}c \\ c \end{pmatrix}$, or $X=c\begin{pmatrix} -\frac{4}{5} \\ 1 \end{pmatrix}=c^*\begin{pmatrix} -4 \\ 5 \end{pmatrix}$, where the notation $c^*=\frac{1}{5}c$ was used for convenience.

Similarly, if $\lambda_2 = 7$, then the equation (9.2) takes the form $(A - 7E) \cdot X = 0$, or $\left(\begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix} - \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}\right) \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$, $\begin{pmatrix} -4 & 4 \\ 5 & -5 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
Hence, one gets the equation $-x_1 + x_2 = 0$, or $x_2 = x_1$. Let $x_1 = c$, then $x_2 = c$, and the eigenvector $X = \begin{pmatrix} c \\ c \end{pmatrix}$, or $X = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

10. The application of vectors in economics

When solving the geometric problems the vector is usually mean the directed line segment. On the contrary, in the economics studies the vector is interpreted as the ordered sequence of numbers that is the algebraic definition of vector. Therefore, the concept of *n*-dimensional vector is widely used in the economics: the vector of products set; vector of prices; vector of cost; vector of expenses. Consider the examples.

Finding the total cost of the goods. Let's assume, that we have *n* different goods. The volume of *i*th product is denoted as $x_i,\ i=1,2,...,n$. Then some set of these products can be written in the form of vector $\vec{x}=\left(x_1,x_2,x_3,...,x_n\right)$, therefore, it is the *n*-dimensional vector. From the economics reasoning one considers only such sets of products, in which $x_i \geq 0$ for any $i=1,2,...,n$. The set of all products vectors is called the space of products *C*.

Let each product has a certain price. All the prices are strictly positive, the price of the single *i*th product is $p_i,\ i=1,2,...,n$. Then the vector $\vec{p}=\left(p_1,p_2,p_3,...,p_n\right)$ is called the vector of prices. The scalar product of these vectors $\vec{p}\cdot\vec{x}=p_1x_1+p_2x_2+...+p_nx_n$ is a number, that determines the total price of the products and is denoted as $c(\vec{x})$. Therefore, **the economic sense of scalar product of vectors is the total price of products**.

Calculation of the price index. Let's denote $\vec{x}$ as the vector of volume of products, $\vec{c}$ as the vector of prices in the current month, $\vec{c}_*$ as the vector of price in the previous month.

Then the price index (%) is calculated by the formula:

$$p=\frac{\vec{c}\cdot\vec{x}}{\vec{c}_*\cdot\vec{x}}\cdot 100\% .$$

The price index shows, on how many percents the prices for the products have changed in the current month in comparison with the previous ones.

11. General conclusions

There exist several definitions of vector:

1) vector is the value which is characterized by a numerical value and a direction
2) vector is a directed line segment (geometric);
3) vector is the ordered collection of the numbers (algebraic).

The geometric vectors are considered on the line, on the plane or in the three-dimensional space.

The algebraic vectors are considered in the spaces of any dimension.

The algebraic notation of vector is comprised of the notation of matrix-row and matrix-column. The equality of vectors, the operations of vector multiplication on a number, the sum, subtraction and scalar product of vectors that is performed in the component form is similar to the operation on matrices.

There exist the linear and non-linear operations on vectors. The linear operations are the addition (subtraction) of vectors and the multiplication of vector on a number. The nonlinear operations contain the vector products. Among them there are:

1) the scalar product of two vectors $\vec{a}$ and $\vec{b}$;
2) the vector product of two vectors $\vec{a}$ and $\vec{b}$;
3) the scalar triple product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

It is important to memorize:

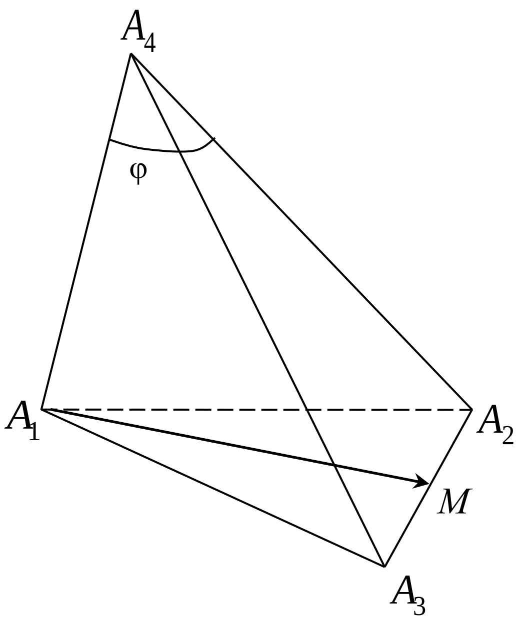
- if the scalar product of two non-zero vectors is equal to zero, then these vectors are perpendicular;
- if the vector product of two non-zero vectors is equal to zero, then these vectors are collinear;
- if the scalar triple product of three non-zero vectors is equal to zero, then these vectors are coplanar.

Example of solving of the problem set

Variant №**

The coordinates of the vertices of the pyramid in the coordinate system $OXYZ$ are given: $A_1(3;10;7)$, $A_2(9;8;6)$, $A_3(2;8;-2)$, $A_4(6;-6;10)$. Then

1) find the components of the vectors $\overrightarrow{A_1A_2}$, $\overrightarrow{A_1A_3}$, $\overrightarrow{A_2A_3}$ which are coincided with the sides of the base of the pyramid;
2) find the components of the vector $\overrightarrow{A_1M}$ where A_1M is a median of the triangle $A_1A_2A_3$;
3) find the direction vector of the vector $\overrightarrow{A_1M}$;
4) find the lengths of the sides of the base $A_1A_2A_3$ of the pyramid;
5) determine the type of triangle $A_1A_2A_3$ (acute, right or obtuse);
6) calculate $\cos\varphi$ where φ is the angle between the sides A_1A_4 and A_2A_4 of the piramid;
7) find the direction cosines of the vector $\overrightarrow{A_1A_4}$ and the normalized vector of $\overrightarrow{A_1A_4}$;
8) verify if the vectors $\overrightarrow{A_1A_2}$ and $\vec{b}=(2;-4;3)$ are collinear;
9) If the vectors $\overrightarrow{A_1A_2}$, $\overrightarrow{A_1A_3}$, $\overrightarrow{A_1A_4}$ are noncoplanar then express the vector $\vec{d}=(2;4;9)$ as a linear combination of this vectors;
10) Calculate the area of the base $A_1A_2A_3$;
11) Find the volume of the pyramid $A_1A_2A_3A_4$.



Solution

1) Calculate the components of the vectors $\overrightarrow{A_1A_2}, \overrightarrow{A_1A_3}, \overrightarrow{A_2A_3}$.

$$\overrightarrow{A_1A_2}=(9-3;8-10;6-7)=(6;-2;-1).$$

$$\overrightarrow{A_1A_3}=(2-3;8-10;-2-7)=(-1;-2;-9).$$

$$\overrightarrow{A_2A_3}=(2-9;8-8;-2-6)=(-7;0;-8).$$

2) According to the triangle rule

$\overrightarrow{A_1A_2}+\overrightarrow{A_2M}=\overrightarrow{A_1M}$. But $\overrightarrow{A_2M}=\frac{1}{2}\overrightarrow{A_2A_3}$.

Then $\overrightarrow{A_1M}=\overrightarrow{A_1A_2}+\frac{1}{2}\overrightarrow{A_2A_3}$. Now write it in components form :

$$\overrightarrow{A_1M}=(6;-2;-1)+\frac{1}{2}(-7;0;-8)=\left(2\frac{1}{2};-2;-5\right).$$

The direction vector of $\overrightarrow{A_1M}$ can be found by dividing $\overrightarrow{A_1M}$ by its length.

3) $\vec{a}_0=\frac{\overrightarrow{A_1M}}{\left|\overrightarrow{A_1M}\right|}$; $\quad\left|\overrightarrow{A_1M}\right|=\sqrt{\left(\frac{5}{2}\right)^2+(-2)^2+(-5)^2}=\sqrt{\frac{25}{4}+4+25}=\sqrt{\frac{25+116}{4}}=\frac{\sqrt{141}}{4}$;

$$\vec{a}_0=\frac{1}{\frac{\sqrt{141}}{2}}\left(\frac{5}{2};-2;-5\right)=\left(\frac{2}{\sqrt{141}}\cdot\frac{5}{2};\frac{2}{\sqrt{141}}\cdot(-2);\frac{2}{\sqrt{141}}\cdot(-5)\right)=\left(\frac{5}{\sqrt{141}};\frac{-4}{\sqrt{141}};\frac{-10}{\sqrt{141}}\right).$$

4) $\quad\left|\overrightarrow{A_1A_2}\right|=\sqrt{6^2+(-2)^2+(-1)^2}=\sqrt{36+4+1}=\sqrt{41}$;

$$\left|\overrightarrow{A_1A_3}\right|=\sqrt{(-1)^2+(-2)^2+(-9)^2}=\sqrt{1+4+81}=\sqrt{86}.$$

$$\left|\overrightarrow{A_2A_3}\right|=\sqrt{(-7)^2+0^2+(-8)^2}=\sqrt{49+64}=\sqrt{113}.$$

5) Find $\cos\angle A_1, \cos\angle A_2, \cos\angle A_3$.

$$\cos\angle A_1=\frac{\overrightarrow{A_1A_2}\cdot\overrightarrow{A_1A_3}}{\left|\overrightarrow{A_1A_2}\right|\cdot\left|\overrightarrow{A_1A_3}\right|}=\frac{6\cdot(-1)+(-2)\cdot(-2)+(-1)\cdot(-9)}{\sqrt{41}\cdot\sqrt{86}}=\frac{-6+4+9}{\sqrt{41}\cdot\sqrt{86}}=$$

$$=\frac{7}{\sqrt{41}\cdot\sqrt{86}}>0.\quad \text{Тому } \angle A_1<\frac{\pi}{2}.$$

$$\cos\angle A_2=\frac{\overrightarrow{A_2A_1}\cdot\overrightarrow{A_2A_3}}{\left|\overrightarrow{A_2A_1}\right|\cdot\left|\overrightarrow{A_1A_3}\right|}=\frac{-\overrightarrow{A_1A_2}\cdot\overrightarrow{A_2A_3}}{\left|\overrightarrow{A_1A_2}\right|\cdot\left|\overrightarrow{A_1A_3}\right|}=\frac{-6\cdot(-7)+2\cdot0+1\cdot(-8)}{\sqrt{41}\cdot\sqrt{113}}=\frac{42-8}{\sqrt{41}\cdot\sqrt{86}}=$$

$$=\frac{34}{\sqrt{41}\cdot\sqrt{86}}>0.\quad \text{Тому } \angle A_2<\frac{\pi}{2}.$$

$$\cos\angle A_3=\frac{\overrightarrow{A_3A_1}\cdot\overrightarrow{A_3A_2}}{\left|\overrightarrow{A_3A_1}\right|\cdot\left|\overrightarrow{A_3A_2}\right|}=\frac{\left(-\overrightarrow{A_1A_3}\right)\cdot\left(-\overrightarrow{A_2A_3}\right)}{\left|\overrightarrow{A_1A_3}\right|\cdot\left|\overrightarrow{A_2A_3}\right|}=\frac{\overrightarrow{A_1A_3}\cdot\overrightarrow{A_2A_3}}{\left|\overrightarrow{A_1A_3}\right|\cdot\left|\overrightarrow{A_2A_3}\right|}=$$

$$=\frac{(-1)\cdot(-7)+(-2)\cdot0+(-9)\cdot(-8)}{\sqrt{86}\cdot\sqrt{113}}=\frac{7+72}{\sqrt{41}\cdot\sqrt{86}}=\frac{79}{\sqrt{41}\cdot\sqrt{86}}>0.\quad \text{Тому } \angle A_3<\frac{\pi}{2}.$$

Therefore, triangle $A_1A_2A_3$ is acute.

6) Use the formula: $\overrightarrow{A_4A_1} = (3-6;10+6;7-10) = (-3;16;-3)$.

$\overrightarrow{A_4A_2} = (6-9;-6-8;10-6) = (-3;-14;4)$.

$\left|\overrightarrow{A_4A_1}\right| = \sqrt{(-3)^2 + 16^2 + (-3)^2} = \sqrt{9+256+9} = \sqrt{274}.$

$\left|\overrightarrow{A_4A_2}\right| = \sqrt{(-3)^2 + (-14)^2 + 4^2} = \sqrt{9+196+14} = \sqrt{219}.$

$$\cos\varphi = \frac{(-3)\cdot(-3)+16\cdot(-14)+(-3)\cdot 4}{\sqrt{274}\cdot\sqrt{219}} = \frac{9-224-12}{\sqrt{274}\cdot\sqrt{219}} = \frac{-227}{\sqrt{274}\cdot\sqrt{219}}.$$

7) The direction cosines of the vector $\overrightarrow{A_1A_4}$ can be found by dividing the coordinates of the vector by its length. Because $\overrightarrow{A_1A_4} = -\overrightarrow{A_4A_1} = (3;-16;3)$ then

$$\cos\alpha = \frac{3}{\sqrt{274}};\ \cos\beta = \frac{-16}{\sqrt{274}};\ \cos\gamma = \frac{3}{\sqrt{274}}.$$

Since the components of the normalized vector of $\overrightarrow{A_1A_4}$ are the direction cosines of this vector, then the normalized(direction) vector of $\overrightarrow{A_1A_4}$ are:

$$\vec{c}_o = \frac{\overrightarrow{A_1A_4}}{\left|\overrightarrow{A_1A_4}\right|} = \left(\frac{3}{\sqrt{274}};-\frac{16}{\sqrt{274}};\frac{3}{\sqrt{274}}\right).$$

8) If the vectors $\overrightarrow{A_1A_2}$ and $\vec{b}$ are collinear then their corresponding components are proportional.

But $\frac{6}{2} \neq \frac{-2}{-4} \neq \frac{-1}{3}$. Therefore, these vectors are noncollinear.

9) First we will prove that vectors $\overrightarrow{A_1A_2}$, $\overrightarrow{A_1A_3}$ i $\overrightarrow{A_1A_4}$ are noncoplanar.

$$\overrightarrow{A_1A_2}\,\overrightarrow{A_1A_3}\,\overrightarrow{A_1A_4} = \begin{vmatrix} 6 & -2 & -1 \\ -1 & -2 & -9 \\ 3 & -16 & 3 \end{vmatrix} = 6\cdot(-2)\cdot 3 + (-1)\cdot(-1)\cdot(-16) + (-2)\cdot(-9)\cdot 3 -$$

$$-3\cdot(-2)\cdot(-1) - (-16)\cdot(-9)\cdot 6 - (-1)\cdot(-2)\cdot 3 = -36-16+54-6-864-6 = -874 \neq 0.$$

Hence, $\overrightarrow{A_1A_2}$, $\overrightarrow{A_1A_3}$, $\overrightarrow{A_1A_4}$ form a basis. Write the vector $\vec{d}$ as a linear combination of this basis vectors:

$$\vec{d} = \alpha \cdot \overrightarrow{A_1A_2} + \beta \cdot \overrightarrow{A_1A_3} + \gamma \cdot \overrightarrow{A_1A_4}, \quad \text{or} \quad \alpha \cdot \begin{pmatrix} -6 \\ -2 \\ 1 \end{pmatrix} + \beta \cdot \begin{pmatrix} -1 \\ -2 \\ -9 \end{pmatrix} + \gamma \cdot \begin{pmatrix} 3 \\ -16 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 9 \end{pmatrix}.$$

In component form we get the following system: $\begin{cases} -6\alpha - \beta + 3\gamma = 2; \\ -2\alpha - 2\beta - 16\gamma = 4; \\ \alpha - 9\beta + 3\gamma = 9. \end{cases}$

Solve the following system using Cramer's Rule.

$\Delta = -874.$

$$\Delta_1 = \begin{vmatrix} 2 & -1 & 3 \\ 4 & -2 & -16 \\ 9 & -9 & 3 \end{vmatrix} = 3 \cdot 2 \cdot \begin{vmatrix} 2 & -1 & 3 \\ 2 & -1 & -8 \\ 3 & -3 & 1 \end{vmatrix} = 6 \cdot (-2 - 18 + 24 + 9 - 48 + 2) = -198.$$

$$\Delta_2 = \begin{vmatrix} 6 & 2 & 3 \\ -2 & 4 & -16 \\ -1 & 9 & 3 \end{vmatrix} = 2 \cdot \begin{vmatrix} 6 & 2 & 3 \\ -1 & 2 & -8 \\ -1 & 9 & 3 \end{vmatrix} = 2 \cdot (36 - 27 + 16 + 6 + 432 + 6) = 938.$$

$$\Delta_3 = \begin{vmatrix} 6 & -1 & 2 \\ -2 & -2 & 4 \\ -1 & -9 & 9 \end{vmatrix} = 2 \cdot \begin{vmatrix} 6 & -1 & 2 \\ -1 & -1 & 2 \\ -1 & -9 & 9 \end{vmatrix} = 2 \cdot (-54 + 18 + 2 - 2 + 108 - 9) = 126.$$

Therefore, $\alpha = \frac{\Delta_1}{\Delta} = \frac{-198}{-874} = \frac{99}{437}$, $\beta = \frac{\Delta_2}{\Delta} = \frac{938}{-874} = \frac{469}{437}$, $\gamma = \frac{\Delta_3}{\Delta} = \frac{126}{-874} = -\frac{63}{437}$ and vector $\vec{d}$ can be expressed as:

$$\vec{d} = \frac{99}{437}\overrightarrow{A_1A_2} + \frac{469}{437}\overrightarrow{A_1A_3} - \frac{63}{437}\overrightarrow{A_1A_4}.$$

10) To find the area of triangle we compute the cross product.

$$S_{\Delta A_1A_2A_3} = \frac{1}{2}\left|\overrightarrow{A_1A_2} \times \overrightarrow{A_1A_3}\right|.$$

$$\overrightarrow{A_1A_2} \times \overrightarrow{A_1A_3} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & -2 & -1 \\ -1 & -2 & -9 \end{vmatrix} = 18\vec{i} - 12\vec{k} + \vec{j} - 2\vec{k} - 2\vec{i} + 54\vec{j} = 16\vec{i} + 55\vec{j} - 14\vec{k}.$$

$\left|\overrightarrow{A_1A_2} \times \overrightarrow{A_1A_3}\right| = \sqrt{16^2 + 55^2 + (-14)^2} = \sqrt{256 + 3025 + 196} = \sqrt{3477}$. Then

$S_{\Delta A_1A_2A_3} = \frac{1}{2}\sqrt{3477} == \frac{\sqrt{3477}}{2}$ (square units).

11) The volume of the tetrahedron $A_1A_2A_3A_4$ can be evaluated by the formula:

$V_{piramid} = \frac{1}{6}\left|\left(\overrightarrow{A_1A_2} \times \cdot \overrightarrow{A_1A_3}\right) \cdot \overrightarrow{A_1A_4}\right|$. Find Triple Scalar product of the vectors.

$$\left(\overrightarrow{A_1A_2} \times \cdot \overrightarrow{A_1A_3}\right) \cdot \overrightarrow{A_1A_4} = \begin{vmatrix} 6 & -2 & -1 \\ -1 & -2 & -9 \\ 3 & -16 & 3 \end{vmatrix} = -874.$$

Therefore, $V_{piramid} = \frac{1}{6} \cdot 874 = \frac{437}{3}$ (cubic units).

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