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PRESENTATION OF A GENERAL 3D SOLUTION OF EQUATIONS OF ELASTICITY THEORY FOR A WIDE CLASS OF ORTHOTROPIC MATERIALS

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Summary. *A mathematical model of the statically loaded three-dimensional orthotropic body was used. The broadest class of orthotropic materials in the Cartesian coordinate system is considered. We find a general representation of the solution of equilibrium equations in displacements for orthotropic materials. The expression of displacements, strains and stresses is obtained through the introduced displacement function, which satisfies the sixth-order equation for partial derivatives.*

Key words: *Cartesian coordinate system, displacement function, orthotropic body, solution of equilibrium equations.*

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Problem statement. The development of science and technology leads to the wide use of structural elements from orthotropic materials in various objects of transport, energy machinery, construction industry and other branches of technology [1–4]. It is possible to predict their mechanical behavior, strength and reliability, after integrating the equations of elasticity theory and finding stresses in the elastic orthotropic body.

Analysis of known research results. Theoretical methods of calculating stresses in orthotropic bodies under the influence of static loads have developed since the middle of the nineteenth century [1–4]. Using the Saint-Venan flow method [2, 3, 5] for some simple loads, the stress state is accurately calculated without building a general three-dimensional solution to the equations of the orthotropic body. Currently, the representation of the general solution is known only for isotropic [6, 7] and transversal-isotropic bodies [8–10], the representation of which differs from each other. It should be noted that the construction of these solutions lasted more than a hundred years. Therefore, determining the analytical type of three-dimensional stresses in the general case of orthotropy is an important task of practical design and materials science [1–4, 10].

Purpose of the work. Construct an expression of the stress state components of three-dimensional elasticity theory for the broadest class of orthotropic materials in the Cartesian coordinate system.

Formulation of the problem and solving equations describing elastic orthotropies in linear elasticity theory. Consider the stress-strained state of a statically loaded three-dimensional orthotropic body. Strain components according to the general Hooke's law related to stress components:

$$\varepsilon_1 = \frac{1}{E_1}\{\sigma_1 - \nu_{21}\sigma_2 - \nu_{31}\sigma_3\},\ \varepsilon_2 = \frac{1}{E_2}\{-\nu_{12}\sigma_1 + \sigma_2 - \nu_{32}\sigma_3\}, \tag{1}$$

$$\varepsilon_3 = \frac{1}{E_3}\{-\nu_{13}\sigma_1 - \nu_{23}\sigma_2 + \sigma_3\}, \; \varepsilon_{kj} = \frac{1}{G_{kj}}\tau_{kj}, \; k \neq j,$$

where the following designations are used: E_k are Young's moduli; ν_{kj} are Poisson's coefficients; G_{kj} are shift moduli of orthotropic material [1, 2]; $\nu_{jk}E_j = \nu_{kj}E_k$, $G_{kj} = G_{jk}$, $\nu_{23}E_2 = \nu_{32}E_3$, $\nu_{23} = \nu_{32}\frac{E_3}{E_2}$. There are nine independent elastic constants that allow different deformative characteristics of orthotropic material in perpendicular directions.

Let's solve relations (1) and find explicit expression of stresses through deformations

$$\sigma_j = \sum_{k=1}^{3} B_{jk}\varepsilon_k, \; \tau_{kj} = G_{kj}\gamma_{kj}, \; k \neq j, \tag{2}$$

where

$$B_{jj} = \frac{(1-\nu_{km}\nu_{mk})E_j}{D}, \; \nu_{kj} = \nu_{jk}\frac{E_j}{E_k},$$

$$B_{jk} = \frac{(\nu_{kj}+\nu_{mj}\nu_{km})E_j}{D}, \quad m \neq k \neq j, \; B_{jk} = B_{kj},$$

$$D = 1-\nu_{23}\nu_{32} - \nu_{12}(\nu_{21}+\nu_{31}\nu_{23}) - \nu_{13}(\nu_{31}+\nu_{32}\nu_{21}) > 0,$$

and components of deformations are expressed through displacements by such ratios:

$$\gamma_{jj} \equiv \varepsilon_j = \frac{\partial u_j}{\partial x_j}, \quad j = \overline{1,3}, \; \gamma_{kj} = \frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k}, \; k, j = \overline{1,3}, \; k \neq j. \tag{3}$$

In the absence of volumetric forces, the solution of problems of linear theory of elasticity of orthotropic body, after using equations of equilibrium and relations (2), (3), is reduced to integration of the following three equations:

$$\begin{aligned}
&L_1 u_1 + \frac{\partial}{\partial x_1}\{D_{12}\frac{\partial}{\partial x_2}u_2 + D_{13}\frac{\partial}{\partial x_3}u_3\} = 0,\\
&L_2 u_2 + \frac{\partial}{\partial x_2}\{D_{12}\frac{\partial}{\partial x_1}u_1 + D_{23}\frac{\partial}{\partial x_3}u_3\} = 0,\\
&\frac{\partial}{\partial x_3}\{D_{13}\frac{\partial}{\partial x_1}u_1 + D_{23}\frac{\partial}{\partial x_2}u_2\} + L_3 u_3 = 0,
\end{aligned} \tag{4}$$

where the operators L_j are

$$L_k = \sum_{j=1}^{3} T_{kj}\frac{\partial^2}{\partial x_j^2}, \; T_{kk} = B_{kk}, \; T_{kj} = G_{kj}, \; D_{kj} = B_{kj} + G_{kj}, \; k \neq j, \; k = \overline{1,3}.$$

Let exclude the displacement from the system (4) in turn u_3, u_2, u_1 and obtain a system of differential equations in partial derivatives

$$\frac{\partial}{\partial x_2}\mathrm{L}_1^1 u_1 - \frac{\partial}{\partial x_1}\mathrm{L}_2^1 u_2 = 0\,,$$
$$\frac{\partial}{\partial x_3}\mathrm{L}_1^1 u_1 - \frac{\partial}{\partial x_1}\mathrm{L}_3^1 u_3 = 0\,, \qquad (5)$$
$$\frac{\partial}{\partial x_3}\mathrm{L}_2^1 u_2 - \frac{\partial}{\partial x_2}\mathrm{L}_3^1 u_3 = 0\,,$$

where

$$\mathrm{L}_{\mathrm{j}}^1 = D_{km}L_j - D_{jk}D_{jm}\frac{\partial^2}{\partial x_j^2}\,,\ k \neq j \neq m\,.$$

For the isotropic case we have

$$B_{jk} = \frac{\nu \mathrm{E}}{(1+\nu)(1-2\nu)} = \lambda\,,\ B_{jj} = \frac{(1-\nu)\mathrm{E}}{(1+\nu)(1-2\nu)} = \lambda + 2\mathrm{G}\,,$$
$$T_{\mathrm{kj}} = G_{\mathrm{kj}} = \frac{\mathrm{E}}{2(1+\nu)} = G\,,\ k \neq j\,,\ \mathrm{L}_{\mathrm{j}}^1 = (1-2\nu)\nabla^2$$

and equation (5) are simplified

$$\Delta\left[\frac{\partial}{\partial x_{\mathrm{k}}}u_{\mathrm{j}} - \frac{\partial}{\partial x_j}u_{\mathrm{k}}\right] = 0\,,\ k \neq j\,.$$

If the operators $\mathrm{L}_{\mathrm{j}}^1$ are not equivalent to each other: $\mathrm{L}_{\mathrm{j}}^1 \neq c\mathrm{L}_{\mathrm{m}}^1$, $j \neq m$, where c is the real number, then the solution of the first equation of the system of equations (5) can be expressed as

$$u_1 = \frac{\partial}{\partial x_1}\mathrm{L}_2^1\varphi_1\,,\ u_2 = \frac{\partial}{\partial x_2}\mathrm{L}_1^1\varphi_1\,, \qquad (6)$$

where φ_1 is an unknown function. Replace the ratio (6) with the second and third equations of the system (5) and get

$$\mathrm{L}_3^1 u_3 = \frac{\partial}{\partial x_3}\mathrm{L}_2^1\mathrm{L}_1^1\varphi_1\,.$$

If we enter a new symbol $\varphi_1 = \mathrm{L}_3^1\varphi$, where φ is an unknown motion function, the solution of the equation system (5) can be written as

$$\mathrm{u}_\mathrm{j} = \prod_{k \neq j} \mathrm{L}_\mathrm{k}^1 \frac{\partial}{\partial x_j}\varphi . \tag{7}$$

Let us put the displacements (7) into the system of equations (4) and find the defining equation for the function of displacements φ

$$\mathrm{L}\varphi = \{\mathrm{L}_1\mathrm{L}_2^1\mathrm{L}_3^1 + D_{12}\frac{\partial^2}{\partial x_2^2}\mathrm{L}_1^1\mathrm{L}_3^1 + D_{13}\frac{\partial^2}{\partial x_3^2}\mathrm{L}_2^1\mathrm{L}_1^1\}\varphi = 0 , \tag{8}$$

where the expression operator L is

$$\begin{gathered} L = \mathrm{d}_1\frac{\partial^6}{\partial x_1^6} + \mathrm{d}_2\frac{\partial^6}{\partial x_1^4\partial x_2^2} + \mathrm{d}_3\frac{\partial^6}{\partial x_1^2\partial x_2^4} + \mathrm{d}_4\frac{\partial^6}{\partial x_1^4\partial x_3^2} + \mathrm{d}_5\frac{\partial^6}{\partial x_1^2\partial x_3^4} + \\ + \mathrm{d}_6\frac{\partial^6}{\partial x_2^6} + \mathrm{d}_7\frac{\partial^6}{\partial x_2^4\partial x_3^2} + \mathrm{d}_8\frac{\partial^6}{\partial x_2^2\partial x_3^4} + \mathrm{d}_9\frac{\partial^6}{\partial x_1^2\partial x_2^2\partial x_3^2} + \mathrm{d}_{10}\frac{\partial^6}{\partial x_3^6}, \end{gathered} \tag{9}$$

and its coefficients have the following meanings:

$$\begin{gathered} \mathrm{d}_1 = \mathrm{T}_{11}\mathrm{T}_{12}\mathrm{T}_{13},\ \mathrm{d}_2 = [\mathrm{T}_{11}(\mathrm{T}_{12}\mathrm{T}_{23} + \mathrm{T}_{13}\mathrm{T}_{22}) + T_{12}^2\mathrm{T}_{13} - D_{12}^2\mathrm{T}_{13}], \\ \mathrm{d}_3 = [\mathrm{T}_{22}(\mathrm{T}_{11}\mathrm{T}_{23} + \mathrm{T}_{12}\mathrm{T}_{13}) + T_{12}^2)\mathrm{T}_{23} - D_{12}^2\mathrm{T}_{23}], \\ \mathrm{d}_4 = [\mathrm{T}_{11}(\mathrm{T}_{12}\mathrm{T}_{33} + \mathrm{T}_{23}\mathrm{T}_{13}) + \mathrm{T}_{12}\mathrm{T}_{13}^2 - D_{13}^2\mathrm{T}_{12}], \\ \mathrm{d}_5 = [(\mathrm{T}_{11}\mathrm{T}_{23} + \mathrm{T}_{13}\mathrm{T}_{12})\mathrm{T}_{33} + \mathrm{T}_{23}\mathrm{T}_{13}^2 - D_{13}^2\mathrm{T}_{23}],\ \mathrm{d}_6 = \mathrm{T}_{12}\mathrm{T}_{22}\mathrm{T}_{23}, \\ \mathrm{d}_7 = [(\mathrm{T}_{13}\mathrm{T}_{23} + \mathrm{T}_{12}\mathrm{T}_{33})\mathrm{T}_{22} + \mathrm{T}_{12}\mathrm{T}_{23}^2 - D_{23}^2\mathrm{T}_{12}], \\ \mathrm{d}_8 = [(\mathrm{T}_{12}\mathrm{T}_{23} + \mathrm{T}_{13}\mathrm{T}_{22})\mathrm{T}_{33} + \mathrm{T}_{13}\mathrm{T}_{23}^2 - D_{23}^2\mathrm{T}_{13}], \\ \mathrm{d}_9 = \{[2D_{12}D_{13}D_{23} - D_{23}^2\mathrm{T}_{11} - D_{12}^2\mathrm{T}_{33} - D_{13}^2\mathrm{T}_{22}] + (\mathrm{T}_{11}\mathrm{T}_{22} + \mathrm{T}_{12}^2)\mathrm{T}_{33} + \\ + \mathrm{T}_{11}\mathrm{T}_{23}^2 + \mathrm{T}_{22}\mathrm{T}_{13}^2 + 2\mathrm{T}_{13}\mathrm{T}_{12}\mathrm{T}_{23}\},\ \mathrm{d}_{10} = \mathrm{T}_{13}\mathrm{T}_{23}\mathrm{T}_{33}. \end{gathered}$$

To describe the stress-strained state of orthotropic bodies, a homogeneous equation is constructed in the sixth order partial derivative (8), (9), which includes all other derivatives $\frac{\partial^2}{\partial x_\mathrm{k}^2}$ for the three coordinate variables. It contains 10 coefficients, which depend in a certain way on 9 independent elastic constants describing orthotropic material. Further simplification of the general kind of equation (8), (9) is significantly difficult because it generally does not decompose into multipliers. We were not able to use the known methods of separation of variables that were used for equations in partial derivatives from two variables.

We use displacements representations (7) and find strain and shear components deformation by formulas (3)

$$\varepsilon_{\mathrm{m}} = \prod_{i\neq m} \mathrm{L}_{\mathrm{i}}^{1} \frac{\partial^2}{\partial x_{\mathrm{m}}^2}\varphi\,,\ m=\overline{1,3}\,,$$
$$\gamma_{ni} = \frac{\partial^2}{\partial x_n \partial x_i}\mathrm{L}_{\mathrm{m}}^{1}(\mathrm{L}_{\mathrm{n}}^{1}+\mathrm{L}_{\mathrm{i}}^{1})\varphi\,,\ m\neq n,i\,,\ n\neq i\,, \tag{10}$$

Set the expression of displacements (7), deformations (10) and by formulas (2) define stress components

$$\sigma_{\mathrm{i}} = \sum_{m=1}^{3} B_{\mathrm{im}} \prod_{n\neq m} \mathrm{L}_{\mathrm{n}}^{1} \frac{\partial^2}{\partial x_{\mathrm{m}}^2}\varphi\,,$$
$$\tau_{kj} = G_{kj}\gamma_{kj}\,,\ k\neq j\,. \tag{11}$$

Knowing the expression of stresses (11) and displacements (7), we record the edge conditions on the surface of the orthotropic body.

Note that the proposed approach and the obtained formulae allow for different orthotropic bodies to set and solve the corresponding edge tasks and to determine their three-dimensional stress-strained state.

Conclusions. It has been established that the general solution of the equations of the elasticity theory of the orthotropic body is expressed through one function that satisfies the equation in sixth-order partial derivatives. Mathematical and physical rigor is maintained in the construction of calculation set of formulas of orthotropic elasticity theory. On the basis of the general solution of the equilibrium equations of the orthotropic body, an expression of deformations and stresses in the Cartesian coordinate system has been built. Obtained results can be used in calculation of stressed state of both thick and thin orthotropic plates, prisms and rods in practical design of structural elements from orthotropic materials.

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ПОДАННЯ ЗАГАЛЬНОГО ТРИВИМІРНОГО РОЗВ'ЯЗКУ РІВНЯНЬ ТЕОРІЇ ПРУЖНОСТІ ДЛЯ ШИРОКОГО КЛАСУ ОРТОТРОПНИХ МАТЕРІАЛІВ

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Резюме. *Присвячено побудові загального розв'язку рівнянь теорії пружності для найширшого класу ортотропних матеріалів у декартовій системі координат. Елементи конструкцій із ортотропних матеріалів широко використовуються в різноманітних об'єктах транспортного, енергетичного машинобудування, будівельній індустрії та інших галузях техніки. Для описування їх напруженого стану використано лінійну математичну модель теорії пружності для статично навантаженого тривимірного ортотропного тіла. Модель включає узагальнений закон Гука, вираз деформацій як через переміщення, так і напруження й рівняння рівноваги пружного тіла в декартовій системі координат. Розроблено методику інтегрування трьох рівнянь рівноваги шляхом почергового виключення переміщень за відсутності об'ємних сил. Знайдено критерії, які має задовольняти найширший клас ортотропних матеріалів, так що можна отримати подання їх однотипного розв'язку у декартовій системі координат. Отримано загальне подання розв'язування рівнянь рівноваги у переміщеннях для ортотропного матеріалу через введену функцію переміщень, яка задовольняє рівняння шостого порядку в частинних похідних від трьох координатних змінних. Воно містить 10 коефіцієнтів, які визначеним способом залежать від 9 незалежних пружних постійних, що описують ортотропний матеріал. Значення десяти коефіцієнтів цього рівняння визначаються через задані модулі Юнга, коефіцієнти Пуассона й модулі зсуву ортотропного матеріалу. Витримано математичну й фізичну строгість при побудові розрахункових формул ортотропної теорії пружності. На основі загального розв'язку рівнянь рівноваги ортотропного тіла записано вираз деформацій і напружень у декартовій системі координат через введену функцію переміщень, яка визначається з розв'язку побудованого рівняння. Отримані результати можуть бути використані в практичному проектуванні елементів конструкцій із ортотропних матеріалів.*

Ключові слова: *декартова система координат, функція переміщень, ортотропне тіло, розв'язок рівнянь рівноваги.*

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