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DEVELOPMENT OF TWO-DIMENSIONAL THEORY OF THICK PLATES BENDING ON THE BASIS OF GENERAL SOLUTION OF LAMÉ EQUATIONS

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Summary. *A theory of bending of the thick plate normally loaded on lateral surfaces, when its stress state is not described by the hypothesis of Kirchhoff–Love or Tymoshenko, is suggested. Its three-dimensional stress-strain state is divided into symmetrical bend and compression. To describe the symmetrical bend, three harmonic functions are used expressing the general solution of the Love equations and three-dimensional stress state of the plate. After integrating the stresses along the plate thickness, bending and torque moments and transverse stresses are expressed through three two-dimensional functions. Closed system of partial differential equations of the eighth order was developed on the introduced two-dimensional functions without the use of hypotheses about the geometric nature of the plate deformation. Three-dimensional boundary conditions are reduced to two-dimensional form.*

Key words*: thick plates, three-dimensional stressed state, stress tensor, Lamé equations.*

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Statement of the problem. Thick plates are widely used in transport, power engineering and civil engineering industries. The development of science and technology puts forward new high demands to the accuracy of investigations of their strength and holding ability. Therefore, there is the need for more complete consideration of the equations and relations of the elasticity theory, while simplifying the initial calculation models, by reducing them to two-dimensional case.

Analysis of the available investigations. Plates with applied bending loads are widely used in building and engineering constructions [1-7]. It is known [1-3] that the thick plates bending should be considered as a three-dimensional problem of the elasticity theory. In paper [8], the new theory of the loaded thick plates was offered on the basis of Kirchhoff-Love theory, taking into account the bending moment gradient, considering that the median surface deflections are significant. It was found [9] that for thick plate bending by transverse force (within the limits of the three-dimensional elasticity theory), the normals to the undeformed median surface significantly deviate from the normal to the deformed and bend. Available plates bending theories [1-8] state the nature of the deformation of the normal to the plate median surface and do not directly take into account the torque applied to the plate contour. In paper [10], torque was taken into account and the plates bending theory was developed on the basis of integration of three-dimensional harmonic equation with the unknown right-hand side.

The objective of the paper is to construct the closed two-dimensional calculation model of thick plates based on the general solution of Love equations and the found representation of three-dimensional stresses, as well as to express moments and transverse forces in the thick plate through three two-dimensional functions satisfying the equations in partial derivatives.

Statement of the problem and development of the outgoing system of equations. Let us consider three-dimensional bending problem of the thick plate with constant thickness h, the plane medial surface of which occupies an area S and coincides with the plane Oxy of the Cartesian coordinate system, where the designation $x_1 = x,\ x_2 = y,\ x_3 = z$ is also introduced. To both of the external plate surfaces ($z = h_j$, $h_1 = h/2$, $h_2 = -h/2$) the normal loads $q_j(x,y)$, $j = \overline{1,2}$ are applied, and tangents are absent. Let us divide the applied loads into two parts. For the first problem, describing the symmetrical plate bending, the normal loads of the flat plate surfaces are equal and directed in one direction:

$$\sigma_z(x,y,h_1) = g^+(x,y),\ \sigma_z(x,y,h_2) = -g^+(x,y), \tag{1}$$

and for the second –? in opposite direction

$$\sigma_z(x,y,\frac{h}{2}) = p^+(x,y), \quad \sigma_z(x,y,-\frac{h}{2}) = p^+(x,y),$$

where $g^+ = \dfrac{1}{2}(q_1 - q_2)$; $p^+ = \dfrac{1}{2}(q_1 + q_2)$; the signs “+”, “–“ describe the functions on the $z = h_1$ and bottom $z = -h_1$ plate surfaces relatively.

Let us consider in detail the symmetric bending of plate, determined by the boundary conditions (1). On the closed lateral surface of the plate Ω the boundary conditions are specified

$$\sigma_n(x,y,z) = \sigma_{1n}|_\Omega,\ \tau_{nt}(x,y,z) = \sigma_{2n}|_\Omega,\ \tau_{nz}(x,y,z) = \sigma_{3n}|_\Omega, \tag{2}$$

where σ_{jn}, $j = \overline{1,3}$ –? are known loads, $\sigma_{jn}(x,y,-z) = -\sigma_{jn}(x,y,z)|_\Omega$, $i = \overline{1,2}$, $\sigma_{3n}(x,y,-z) = \sigma_{3n}(x,y,z)|_\Omega$.

For the solution of the boundary value problem (1), (2) we use the general representation of the solution of the Love equations given in [11]

$$u_x = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}, \quad u_y = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}, \quad u_z = \frac{\partial P}{\partial z} - 4(1-\nu)\Phi, \tag{3}$$

where $P = z\Phi + \Psi$; Φ, Ψ, Q – are - three-dimensional harmonic functions of displacements; ν – are Poisson ratio. Let us use the displacements (3) and write the expression of the normal stresses

$$\sigma_j = 2G\left[\frac{\partial^2 P}{\partial x_j^2} - 2\nu\frac{\partial \Phi}{\partial x_3} - (-1)^j\frac{\partial^2 Q}{\partial x_1 \partial x_2}\right],\ \sigma_3 = 2G\left[\frac{\partial^2 P}{\partial x_3^2} - 2(2-\nu)\frac{\partial \Phi}{\partial x_3}\right] \tag{4}$$

and for the tangential stresses

$$\tau_{12} = G\left[2\frac{\partial^2 P}{\partial x_1 \partial x_2} + \frac{\partial^2 Q}{\partial x_2^2} - \frac{\partial^2 Q}{\partial x_1^2}\right],$$

$$\tau_{j3} = G\left[\frac{\partial}{\partial x_j}\left[2\frac{\partial P}{\partial x_3} - 4(1-\nu)\Phi\right] - (-1)^j\frac{\partial^2 Q}{\partial x_{3-j}\partial x_3}\right], \quad j = \overline{1,2}, \tag{5}$$

where $G=\dfrac{E}{2(1+\nu)}$, E are shear and Young's moduli, respectively. The biharmonic function P satisfies the following equation:

$$\Delta P+\frac{\partial^2}{\partial z^2}P=2\frac{\partial}{\partial z}\Phi\ , \tag{6}$$

where $\Delta=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}$ is the two-dimensional Laplace operator.

It follows from the relations (1), (4), (5) that for this load the functions $P,\ \Psi,\ Q$ are odd relatively to the variable z, and the function Φ is even. From the conditions (1), (2) and the symmetry of the introduced functions we derive the following dependencies:

$$u_i(x,y,-z)=-u_i(x,y,z)\,,\ i=\overline{1,2}\,,\ \ u_3(x,y,-z)=u_3(x,y,z)\,,$$
$$\frac{\partial P^-}{\partial z}=\frac{\partial P^+}{\partial z}\,,\ \frac{\partial \Psi^-}{\partial z}=\frac{\partial \Psi^+}{\partial z}\,,\ \frac{\partial Q^-}{\partial z}=\frac{\partial Q^+}{\partial z}\,,\ \Phi^-=\Phi^+\,, \tag{7}$$

where u_i are the displacements in the direction of the corresponding axes of the Cartesian coordinate system.

Let us develop two-dimensional theory of bending of the thick plate. In order to do this, we substitute the found three-dimensional stresses (4), (5) in the known expressions [1, 2, 4] of the moments and transverse forces and get:

$$M_j(x_1,x_2)=\int_{-h_1}^{h_1}z\sigma_j dz=2G\left[\frac{\partial^2 P_1}{\partial x_j^2}-2\nu\psi-(-1)^j\frac{\partial^2 Q_1}{\partial x_1\partial x_2}\right],$$
$$H(x_1,x_2)=\int_{-h_1}^{h_1}z\tau_{12}dz=G\left[2\frac{\partial^2 P_1}{\partial x_1\partial x_2}+\frac{\partial^2 Q_1}{\partial x_2^2}-\frac{\partial^2 Q_1}{\partial x_1^2}\right], \tag{8}$$
$$N_j(x_1,x_2)=2G\left[2\frac{\partial}{\partial x_j}[P^+-(1-\nu)\tilde{\Phi}]-(-1)^j\frac{\partial Q^+}{\partial x_{3-j}}\right],\quad j=\overline{1,2}\,,$$

where the introduced two-dimensional functions, equal to the integrals $P_1=\int_{-h_1}^{h_1}zPdz$, $Q_1=\int_{-h_1}^{h_1}zQdz$, $\tilde{\Phi}=\int_{-h_1}^{h_1}\Phi dz$ are denoted and used,

$$\psi=\int_{-h_1}^{h_1}z\frac{\partial}{\partial z}\Phi dz=h\Phi^+-\tilde{\Phi}\ . \tag{9}$$

From the condition of harmony of the displacement functions and relation (6), we derive the equations that make a connection between the introduced two-dimensional functions

$$\Delta P_1 = -h\frac{\partial}{\partial z}P^+ + 2P^+ + 2\psi\,,\quad \Delta\tilde{\Phi} = -2\frac{\partial}{\partial z}\Phi^+\,,\quad \Delta Q_1 = -h\frac{\partial}{\partial z}Q^+ + 2Q^+\,. \tag{10}$$

Here is the equations of the plate equilibrium under its bending [1, 2]:

$$N_1 = \frac{\partial M_1}{\partial x} + \frac{\partial H}{\partial y}\,,\quad N_2 = \frac{\partial M_2}{\partial y} + \frac{\partial H}{\partial x}\,,$$

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_2}{\partial y} + 2g^+ = 0\,,\quad \frac{\partial^2 M_1}{\partial x^2} + \frac{\partial^2 M_2}{\partial y^2} + 2\frac{\partial^2 H}{\partial x \partial y} + 2g^+ = 0\,. \tag{11}$$

Let us use in the first two equations (11) the found moments (8) and determine the transverse forces N_j, $j = \overline{1,2}$ due to introduced two-dimensional functions

$$N_j = 2G\{\frac{\partial}{\partial x_j}[\Delta P_1 - 2\nu\psi] - (-1)^j \frac{1}{2}\frac{\partial}{\partial x_{3-j}}\Delta Q_1\}\,. \tag{12}$$

Hence, all efforts and moments are expressed in terms of three functions: P_1, Q_1 and ψ.

Such defining relation between the introduced functions follows from the equations of equilibrium (11) and relations (12)

$$\Delta\Delta P_1 = 2\nu\Delta\psi - g^+ / G\,. \tag{13}$$

Using the formulas (4), (5), we express the boundary conditions (1) and the conditions for the absence of tangential loads on the side plate surfaces in the following form:

$$\frac{\partial^2 P^+}{\partial x_3^2} - 2(2-\nu)\frac{\partial \Phi^+}{\partial x_3} = \frac{1}{2G}g^+(x,y)\,, \tag{14}$$

$$\frac{\partial}{\partial x_j}\left[2\frac{\partial P^+}{\partial x_3} - 4(1-\nu)\Phi^+\right] = (-1)^j \frac{\partial^2 Q^+}{\partial x_{3-j}\partial x_3}\,,\quad j = \overline{1,2}\,. \tag{15}$$

From the equations (15) after simple transformations, we obtain the following harmonic conditions:

$$\Delta[\frac{\partial P^+}{\partial x_3} - 2(1-\nu)\Phi^+] = 0\,,\quad \Delta\frac{\partial Q^+}{\partial x_3} = 0\,. \tag{16}$$

We use paper [10] to define the function ψ and write the normal displacements of the external surface of the plate

$$u_z^+ = \frac{\partial P^+}{\partial z} - 4(1-\nu)\Phi^+\,.$$

Let us find the mean value of the displacement u_z along the axis Oz

$$\tilde{u}_z = \frac{1}{h}\int_{-h_1}^{h_1} u_z dz = \frac{1}{h}(2P^+ - 4(1-\nu)\tilde{\Phi}) .$$

Let us assume that it approximately equals u_z^+. After transformations we get

$$4(1-\nu)\psi = h\frac{\partial P^+}{\partial z} - 2P^+ . \qquad (17)$$

We use the relation (17) in the first equation (10) and find the determining dependence of the plates bending theory

$$\Delta P_1 = -2(1-2\nu)\psi . \qquad (18)$$

Taking into account relation (13) we obtain the basic equation of bending theory of thick plates

$$\Delta\Delta P_1 = -\frac{1-2\nu}{(1-\nu)G} g^+ . \qquad (19)$$

Let us assume that the function Q^+ is harmonious. The biharmonicity condition follows from equations (10), (16)

$$\Delta\Delta Q_1 = 0 . \qquad (20)$$

The evident integration of the system of equations (18) – (20) taking into account the boundary conditions (14), (15) on the external plate surfaces, in the general case of load $g^+ \neq 0$, is the complicated mathematical problem.

Simplification of the bending theory when the load is $g^+ = 0$, and unpaired relatively to the median surface normal loads are applied on the edges of the plate. In this case, it follows from equation (19) that the function P_1 is biharmonic, and from equation (13) that the function ψ is harmonic.

Taking into account the above mentioned, we provide the required functions

$$\psi = -\frac{\partial\varphi_1(x,y)}{\partial y} , \qquad (21)$$

$$P_1 = (1-2\nu)y\varphi_1 + g_1, \quad Q_1 = 2y\frac{\partial\varphi_2}{\partial x} + g_2(x,y) , \qquad (22)$$

where φ_j, g_j – are harmonic functions. In the general case the functions g_j, $j = \overline{1,2}$ can be expressed in terms of two functions

$$g_1 = [\frac{\partial\phi}{\partial x} - \frac{\partial f}{\partial x}], \quad g_2 = [\frac{\partial\phi}{\partial y} + \frac{\partial f}{\partial y}] , \qquad (23)$$

where ϕ, f – are harmonic functions. Taking into account the expression of functions (21) – (23), we express the moments (8) in the following way:

$$M_1 = 2G\left[(1-2\nu)y\frac{\partial^2\varphi_1}{\partial x^2} + 2(\nu\frac{\partial\varphi_1}{\partial y} + \frac{\partial^2\varphi_2}{\partial x^2} + y\frac{\partial^3\varphi_2}{\partial x^2\partial y} + \frac{\partial^3 f}{\partial^2 y\partial x})\right],$$

$$M_2 = 2G\left[(1-2\nu)y\frac{\partial^2\varphi_1}{\partial y^2} + 2(1-\nu)\frac{\partial\varphi_1}{\partial y} - 2(\frac{\partial^2\varphi_2}{\partial x^2} + y\frac{\partial^3\varphi_2}{\partial x^2\partial y} + \frac{\partial^3 f}{\partial^2 y\partial x})\right], \tag{24}$$

$$H = 2G\left[(1-2\nu)(\frac{\partial\varphi_1}{\partial x} + y\frac{\partial^2\varphi_1}{\partial x\partial y}) + 2[y\frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial y}]\frac{\partial\varphi_2}{\partial x} - 2\frac{\partial^3 f}{\partial^2 x\partial y}\right].$$

The relations (23) identically satisfy the equation of equilibrium (11). From conditions (11) we express the transverse forces

$$N_j = 4G\left[(1-\nu)\frac{\partial}{\partial x_j}\frac{\partial\varphi_1(x,y)}{\partial y} - (-1)^j\frac{\partial}{\partial x_{3-j}}\frac{\partial^2\varphi_2}{\partial x\partial y}\right], \quad j = \overline{1,2}. \tag{25}$$

It follows from dependences (24), (25) that the function ϕ does not affect the moments and transverse forces in the plate. Thus it can be neglected.

The obtained relations allow us to reduce the three-dimensional boundary conditions (2) given to the side surface of the plate Ω to the conditions on its midline L by integrating the thickness of the plate [1] and using the relations (24), (25):

$$\begin{gathered} M_1\sin^2\theta + M_2\cos^2\theta + H\sin 2\theta = M_g\,|_L, \\ \frac{1}{2}(M_2 - M_1)\sin 2\theta - H\cos 2\theta = H_g\,|_L, \\ N_2\cos\theta + N_1\sin\theta = N_g\,|_L, \end{gathered} \tag{26}$$

where $M_g = \int_{-h_1}^{h_1} z\sigma_{1n}dz\,|_\Omega$, $H_g = \int_{-h_1}^{h_1} z\sigma_{2n}dz\,|_\Omega$ − are - external bending and torque moments, $N_g = \int_{-h_1}^{h_1}\sigma_{3n}dz\,|_\Omega$ − is the transverse force, θ − is the angle between the axis Oy and the normal to the contour.

Let us assume that the boundary value problem (21), (26), (26) is solved and the distributed moments and transverse forces are found, through which the stresses on the plate surface are determined. If we know these stresses, we will determine the deformation and displacement of the surfaces of the plate.

Conclusions. Two-dimensional theory of symmetrical bending of the thick plate is developed on the basis of the general solution of the Lamé equations, without using the hypotheses about the distribution of displacements and stresses. Moments and transverse forces are expressed through two biharmonic functions with known right-hand sides. The theory of thick plates bending, which evidently takes into account the torque and satisfies the torque moments and transverse forces specified along the curvilinear plate contour is offered. The obtained results can be used in calculating the stressed state of thick plates.

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ПОБУДОВА ДВОВИМІРНОЇ ТЕОРІЇ ЗГИНУ ТОВСТИХ ПЛАСТИН НА ОСНОВІ ЗАГАЛЬНОГО РОЗВ’ЯЗКУ РІВНЯНЬ ЛЯМЕ

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Резюме. *Запропоновано теорію згину товстої пластини, нормально навантаженої на бічних поверхнях, коли її напружений стан не описують гіпотези Кірхгофа-Лява або Тимошенка. Її тривимірний напружено-деформований стан розділено на симетричні згин і стиск. Для опису симетричного згину використано три гармонічних функції, які виражають загальний розв’язок рівнянь Ляме й описують тривимірний напружений стан пластини. Після інтегрування напружень по товщині пластини виражено згинальні та крутні моменти й поперечні зусилля через три двовимірні функції. Побудовано замкнуту систему рівнянь у часткових похідних восьмого порядку на введені двовимірні функції без використання гіпотез про геометричний характер деформування пластини. Тривимірні крайові умови зведені до двовимірного вигляду.*

Ключові слова: *товсті пластини, тривимірний напружений стан, тензор напружень, рівняння Ляме.*

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