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AN EXACT SERIES SOLUTION FOR FREE VIBRATION OF CYLINDRICAL SHELL WITH ARBITRARY BOUNDARY CONDITIONS

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Summary. Simple accurate formulas for the natural frequencies of circular cylindrical shells are presented for modes in which transverse deflection dominates. Based on the Donnell-Mushtari thin shell theory the equations of motion of the circular cylindrical shell are introduced, using series expansion for axial coordinate and Fourier series for the circumferential direction, a simple explicit solution is obtained. Also, the influence of deformation component is investigated, it is shown that it can be neglected. Good agreement with experimental data and FEM is shown. The advantage of a current approach over the existing formulas is simplicity in programming.

Key words: Cylindrical shell, domain decomposition method, natural frequencies, free vibration.

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Nomenclature

\( R, h, l \)  
Mean radius; Shell thickness; Shell length;

\( \rho, E, \mu \)  
Density of shell material; Young's modulus; Poisson ratio;

\( N_x, N_\phi \)  
Axial and circumferential normal forces;

\( Q_x, Q_\phi \)  
Axial and circumferential bending forces;

\( L \)  
Tangential force;

\( M_x, M_\phi \)  
Axial and circumferential moments;

\( M_\tau \)  
Tangential moment;

\( u, v, w \)  
Axial, circumferential and radial displacements;

\( \varepsilon_x, \varepsilon_\phi, \varepsilon_\tau \)  
Median surface strains;

\( \chi_x, \chi_\phi, \chi_\tau \)  
Bending strains;

\( n, m \)  
Wave number in circumferential and axial direction
**Introduction.** Cylindrical shells are the most investigated type of shells for which there are extensive experimental data and many analytical solutions have been obtained. To solve the shell equations expansion in Fourier series \( \sum_{n} \cos(n\varphi) + \sin(n\varphi) \) along the circumferential coordinate \( \varphi \) is usually used, and then various methods for solving the obtained ordinary differential equation of the 8th degree is suggested. Nowadays, a large number of analytical and numerical methods for solving the problem of free and forced vibrations of an isotropic and composite cylindrical shell have been proposed: exact analytical solutions [1], the Rayleigh-Ritz method [2], the Bubnov-Galerkin method [3, 4], the domain decomposition approach [5], wave propagation method [6].

At the same time, the rapid development of universal computer programs based on the finite element method (FEM) raises the question of the need for further analytical studies. The problem with analytical methods lies in the fact that they aren’t simple in realization, in proving their accuracy and further applications[1, 5]. On other hand, FEM solution is also difficult to analyze, it does not provide the clear engineering understanding of the range of natural frequencies and shape of natural forms. From an engineering point of view, the existence of simple formulas for estimating the natural frequencies of vibration is essential: firstly, for solving the problems of shell dynamic analysis [7], and secondly for testing and analyzing problems solved using FEM. The presence of such formulas also makes it possible to quickly estimate the frequency spectrum of a structure and, if necessary, carry out a refined calculation using an FEM.

The greatest interest in "engineering formulas" was in the 60-70s, when the first approximation expressions were proposed on the basis of the Bubnov-Galerkin method [3, 4]. Perhaps the best engineering estimates with the help of V. Vlasov's hypotheses and variational principles are obtained in the book of Kan [8]. The disadvantage of his solution is the incorrect application of hypotheses, as well as the absence of dependencies for all conditions of supporting the shell. Recent work in this area can be noted for membrane approximation [1], and a comparative analysis of formulas [9]. It should be noted that accurate formulas without any hypotheses and assumptions have a practical value if they are rather simple and understandably written.

The derivation of simple and accurate formulas for the natural frequencies of a cylindrical shell for various boundary conditions, including elastically supported edges, written explicitly is the main goal of this paper. The effect of the deformation component on natural frequencies is also analyzed and a comparison with the experimental data and the results of other researchers is given.

**Equations of motion.** Initial system of dynamic equations for the thin shell:

\[
\frac{\partial N_x}{\partial x} + \frac{\partial L}{R \partial \varphi} + \rho \ddot{u} = 0, \quad (1a)
\]

\[
\frac{\partial N_\varphi}{R \partial \varphi} + \frac{\partial L}{\partial x} + Q_\varphi + \rho \ddot{v} = 0, \quad (1b)
\]

\[
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_\varphi}{R \partial \varphi} - \frac{N_\varphi}{R} + \rho \ddot{w} = 0, \quad (1c)
\]

\[
Q_\varphi = \frac{\partial M_\varphi}{R \partial \varphi} + \frac{\partial M_x}{\partial x}, \quad (1d)
\]

\[
Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_\varphi}{R \partial \varphi}, \quad (1e)
\]

Inner forces are connected with deformations using physical equations:
\[ N_{\varphi} = -H \left( \varepsilon_{\varphi} + \mu \varepsilon_x \right), \tag{2a} \]
\[ N_x = -H \left( \varepsilon_x + \mu \varepsilon_{\varphi} \right), \tag{2b} \]
\[ L = -G \varepsilon_{x\varphi} h, \tag{2c} \]
\[ M_x = -H \delta \left( \chi_x + \mu \chi_{\varphi} \right), \tag{2d} \]
\[ M_{\varphi} = -H \delta \left( \chi_{\varphi} + \mu \chi_x \right), \tag{2e} \]
\[ M_{x\varphi} = -\frac{H \delta}{2} \left( 1 - \mu \right) \chi_{x\varphi}, \tag{2f} \]

Here we used notifications: \[ H = \frac{Eh}{1 - \mu^2}, \quad \delta = \frac{h^2}{12}. \]

Geometrical equations combine displacements with strains:
\[ \varepsilon_x = \frac{\partial u}{\partial x}, \tag{3a} \]
\[ \varepsilon_{\varphi} = \frac{1}{R} \frac{\partial v}{\partial \varphi} + \frac{w}{R}, \tag{3b} \]
\[ \varepsilon_{x\varphi} = \frac{1}{R} \frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial x}, \tag{3c} \]
\[ \gamma_x = \frac{\partial w}{\partial x}, \tag{3d} \]
\[ \gamma_{\varphi} = -\frac{\partial w}{\partial x} + \frac{v}{R}. \tag{3e} \]

Bending strains depend on displacements:

\[ \chi_x = -\frac{\partial^2 w}{\partial x^2}, \tag{4a} \]
\[ \chi_{\varphi} = -\frac{\partial^2 w}{\partial \varphi^2}, \tag{4b} \]
\[ \chi_{x\varphi} = \frac{\partial v}{R \partial x} - \frac{2 \partial^2 w}{R \partial x \partial \varphi}, \tag{4c} \]

The system of equations given above is sufficient for solving the problem of free vibration of a cylindrical shell.

**Numerical solution.** Consider a numerical procedure for solving eighth-order differential equation, which follows from the governing Eqs. (1)–(4). This solution is the most accurate as it does not use essential simplifications at the stage of the problem statement. As the initial unknowns we use the expansions of eight parameters in trigonometric series, namely: the longitudinal displacement \( u(x, \varphi, t) \), tangential displacement \( v(x, \varphi, t) \), radial displacement \( w(x, \varphi, t) \), rotation angle \( \gamma_x(x, \varphi, t) \), axial stress resultant \( N_x(x, \varphi, t) \), shearing stress resultant \( L(x, \varphi, t) \), transverse stress resultant \( Q_x(x, \varphi, t) \), and couple \( M_x(x, \varphi, t) \).

From Eqs.(1a-1d)
\[ \frac{\partial N_x}{\partial x} = -\frac{\partial L}{R \partial \varphi} - \rho h \frac{\partial^2 u}{\partial t^2}, \tag{5a} \]
An exact series solution for free vibration of cylindrical shell with arbitrary boundary conditions

\[
\frac{\partial L}{\partial x} = -\frac{\partial N_\phi}{R \partial \phi} - \frac{O_\phi}{R} - \rho h \frac{\partial^2 v}{\partial t^2}, \quad (5b)
\]

\[
\frac{\partial Q_x}{\partial x} = -\frac{\partial Q_\phi}{R \partial \phi} + \frac{N_\phi}{R} - \rho h \frac{\partial^2 w}{\partial t^2}, \quad (5c)
\]

\[
\frac{\partial M_x}{\partial x} = Q_x - \frac{\partial M_{x\phi}}{R \partial \phi}, \quad (5d)
\]

From Eqs.(2a), (2b) and (3a)

\[
\frac{\partial u}{\partial x} = -\frac{N_x - \mu N_\phi}{E h}, \quad (5e)
\]

From Eqs.(2c) and (3c)

\[
\frac{\partial v}{\partial x} = \epsilon_{x\phi} - \frac{1}{R} \frac{\partial u}{\partial \phi} = -\frac{L}{hG} - \frac{1}{R} \frac{\partial u}{\partial \phi}, \quad (5f)
\]

From Eq. (3d):

\[
\frac{\partial w}{\partial x} = -\gamma_x, \quad (5g)
\]

From Eqs. (4a), (3d) and (2d,2e):

\[
\frac{\partial \gamma_x}{\partial x} = \chi_x = -\frac{M_x - \mu M_\phi}{E h \delta}, \quad (5h)
\]

The quantities \( N_\phi, Q_\phi, M_\phi, \) and \( M_{x\phi}, \) which can be expressed through the accepted unknown parameters, appear in Eqs. (5). We obtain the expression for \( N_\phi \) by excluding \( \epsilon_\phi \) from the physical equations for forces (2a-2d):

\[
N_\phi = \mu N_x - \epsilon_\phi E h = \mu N_x - \frac{E h}{R} \left( \frac{\partial v}{\partial \phi} + w \right). \quad (6a)
\]

The expression for \( M_\phi \) entering this equation is written according to the physical equation (2e) and, with the account of the expression for \( \chi_\phi \) (4b), has the following form:

\[
M_\phi = \mu M_x - \chi_\phi E h \delta = \mu M_x - \frac{E h \delta}{R^2} \left( \frac{\partial v}{\partial \phi} - \frac{\partial^2 w}{\partial \phi^2} \right), \quad (6b)
\]

\[
M_\phi = \mu M_x - \chi_\phi E h \delta = \mu M_x + \frac{E h \delta}{R^2} \left( w + \frac{\partial^2 w}{\partial \phi^2} \right). \quad (6b')
\]

Here and below equations with «'» denote accounting for deformation component.

Eq. (2f) serves for the determination of \( M_{x\phi} \). By substituting the expression for \( \chi_{x\phi} \) (4c) into this equation, we have:

\[
M_{x\phi} = -\frac{H \delta (1 - \mu)}{2R} \left( \frac{\partial v}{\partial x} - 2 \frac{\partial^2 w}{\partial \phi \partial x} \right) = \frac{H \delta (1 - \mu)}{2R} \left( \frac{L}{hG} + \frac{1}{R} \frac{\partial u}{\partial \phi} - 2 \frac{\partial \gamma_x}{\partial \phi} \right), \quad (6c)
\]

\[
M_{x\phi} = -\frac{H \delta (1 - \mu)}{2R} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{R \partial \phi} \right) - 2 \frac{\partial^2 w}{\partial \phi \partial x} = \frac{H \delta (1 - \mu)}{2R} \left( \frac{L}{hG} + \frac{2}{R} \frac{\partial u}{\partial \phi} - 2 \frac{\partial \gamma_x}{\partial \phi} \right). \quad (6c')
\]

The fifth equilibrium equation (1e) serves to determine \( Q_\phi \).
\[ Q_\phi = \frac{\partial M_\phi}{\partial \phi} + \frac{\partial M_{x\phi}}{\partial x}. \] (6d)

The unknown functions are written in the following way:

\[ N_x(x, \phi, t) = n_x(x) \cos(n \phi) \sin(\omega t), \] (7a)

\[ L(x, \phi, t) = l(x) \sin(n \phi) \sin(\omega t), \] (7b)

\[ Q_x(x, \phi, t) = q_x(x) \cos(n \phi) \sin(\omega t), \] (7c)

\[ M_x(x, \phi, t) = \frac{\delta}{R} m_x(x) \cos(n \phi) \sin(\omega t), \] (7d)

\[ u(x, \phi, t) = \frac{R}{E_h} u(x) \cos(n \phi) \sin(\omega t), \] (7e)

\[ v(x, \phi, t) = \frac{R}{E_h} v(x) \sin(n \phi) \sin(\omega t), \] (7f)

\[ w(x, \phi, t) = \frac{R}{E_h} w(x) \cos(n \phi) \sin(\omega t), \] (7g)

\[ \gamma_x(x, \phi, t) = \frac{1}{E_h} \gamma_x(x) \cos(n \phi) \sin(\omega t). \] (7h)

We write a system of eight ordinary deferential equations in terms of main variables:

\[ \frac{dn_x(x)}{dx} = \frac{n}{R} I(x) - \Omega^2 u(x), \] (8a)

\[ \frac{dl(x)}{dx} = \left( \frac{\delta}{R^2} - \mu \right) n \frac{n_x(x)}{R} - \frac{n \delta}{R^3} m_x(x) + \left( \frac{n^2}{R} - \frac{\delta}{R^2} \Omega^2 \right) v(x) + \left( \frac{n}{R} + \frac{\delta n (n^2 - 1)}{R^3 (1 + \mu)} \right) w(x), \] (8b)

\[ \frac{dq_x(x)}{dx} = \left( \frac{n^2}{R^2} - \mu \right) \frac{n_x(x)}{R} - \frac{n \delta}{R^3} m_x(x) + \left( \frac{n^2}{R} + \frac{\Omega^2 \delta n}{R^2} \right) v(x) + \left( \frac{1}{R} - \Omega^2 + \frac{\delta n^2 (n^2 - 1)}{R^3 (1 + \mu)} \right) w(x), \] (8c)

\[ \frac{dm_x(x)}{dx} = \frac{n}{R} l(x) - q_x(x) - \frac{1}{2} \frac{n^2}{(1 + \mu) R} u(x) + \frac{n^2}{(1 + \mu) R} \gamma_x(x), \] (8d)

\[ \frac{du(x)}{dx} = \frac{1 - \mu^2}{R} n_x(x) + \mu \frac{n}{R} v(x) + \frac{1}{R} w(x), \] (8e)

\[ \frac{dv(x)}{dx} = 2 \frac{1 + \mu}{R} l(x) - \frac{n}{R} u(x), \] (8f)

\[ \frac{dv(x)}{dx} = \frac{1}{R} \gamma_x(x), \] (8g)

\[ \frac{d\gamma_x(x)}{dx} = \frac{1 - \mu^2}{R} m_x(x) + \mu \frac{n}{R} v(x) + \frac{n^2}{R} w(x). \] (8h)

Here we used notation: \( \Omega^2 = \omega^2 \frac{\rho R}{E} \). The solution of Eqs. (8) can be easily found by series expansion, i.a. assuming that solution is:

\[ n_x(x) = n_{x_0} + C_{11} \cdot x + C_{12} \cdot x^2 + C_{13} \cdot x^3 + \ldots \] (9)

\[ \gamma_x(x) = \gamma_{x_0} + C_{81} \cdot x + C_{82} \cdot x^2 + C_{83} \cdot x^3 + \ldots. \] (9)

\( C_{11}, C_{12}, \ldots C_{82}, C_{83} \) - constants defined from Eqs. (9). We can use only several first terms in solution (9), for example accounting for coefficients of expansion \( x^0, x^1, x^2, x^3 \) gives a reliable accuracy. The coefficient matrix for \( x^0 \) is matrix of ones, and for \( x^1 \):
The proposed solution is easily programmed using the method of initial parameters. In this case, the shell is divided into small sections, which makes it possible not to increase the number of terms in the polynomial expansion (9). To complete the formulation of the problem, it is also necessary to address the boundary conditions.

**Boundary conditions.** At each edge of the shell, one of 16 types of homogeneous boundary conditions can be specified, they are determined by all possible combinations of the following four equations:

\[
\begin{align*}
&w = 0 \quad \text{or} \quad Q_x + \frac{\partial M_{x\varphi}}{\partial \varphi} = 0, \\
&\gamma_x = 0 \quad \text{or} \quad M_x = 0, \\
&u = 0 \quad \text{or} \quad N_x = 0, \\
&v = 0 \quad \text{or} \quad L - \frac{1}{R} M_{x\varphi} = 0.
\end{align*}
\]

From Eq.(10) boundary conditions for an elastically restrained shell can be easily obtained:

\[
\begin{align*}
&w \cdot k_w = Q_x + \frac{\partial M_{x\varphi}}{\partial \varphi}, \\
&\gamma_x \cdot k_\gamma = M_x, \\
&u \cdot k_u = N_x, \\
&v \cdot k_v = L - \frac{1}{R} M_{x\varphi}.
\end{align*}
\]

Here \(k_u, k_v, k_w, k_\gamma\) are axial, circumferential, radial, rotational spring stiffnesses.

Note that the most common boundary conditions are the simple support (the Navier boundary condition):

\[
\begin{align*}
&w = 0, \quad M_x = 0, \quad N_x = 0, \quad v = 0. \\
&\text{Clamped edge:} \quad w = 0, \quad \gamma_x = 0, \quad u = 0, \quad v = 0. \\
&\text{Free edge:} \quad Q_x + \frac{\partial M_{x\varphi}}{\partial \varphi} = 0, \quad M_x = 0, \quad N_x = 0, \quad L - \frac{1}{R} M_{x\varphi} = 0.
\end{align*}
\]
To use Eq.(19a) and (19d), the boundary conditions \( Q_x + \frac{\partial M_{sp}}{\partial \varphi} = 0 \) and \( L - \frac{1}{R} M_{sp} = 0 \) should be rewritten in terms of the main variables:

\[
Q_x + \frac{\partial}{\partial \varphi} \left( \frac{H\delta(1-\mu)}{2R} \left( \frac{L}{hG} + \frac{1}{R} \frac{\partial u}{\partial \varphi} - 2 \frac{\partial \gamma_x}{\partial \varphi} \right) \right) = 0 ,
\]

(13a)

\[
L - \frac{H\delta(1-\mu)}{2R^2} \left( \frac{L}{hG} + \frac{1}{R} \frac{\partial u}{\partial \varphi} - 2 \frac{\partial \gamma_x}{\partial \varphi} \right) = 0 .
\]

(13b)

Thus, we can use Eq.(13a) and (13b) directly in our numeric scheme.

**Results and discussion.** To illustrate how the frequency equation agrees with reality, its results are compared with experimental data[10]. The results of comparison are shown in Table 1, natural frequencies according to exact formulas of Xing[1], Smith[11] and Cammalleri [12], are also present. It should be noted that our results are obtain for two options: with deformation component and without it, the differences between them is negligible, thus only one value is present in Table 1. The obtained results allow us to state that our formula is in good agreement with the experimental data, as well as with other exact solutions, such as Xing [1], Smith [11] and Cammalleri [12]. Note, that Cammalleri [12] approach is less accurate, as far as it has some simplifying assumptions. The natural frequencies calculated using ANSYS are also presented, the results of which also converge well.

**Table 1**

Natural frequencies of a clamped shell:

\( l=305\text{mm}, R=76\text{mm}, h=0.254\text{mm}, E=207\text{GPa}, \rho=7833\text{kg/m}^3, \mu=0.3 \)

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Fig. 1 and Fig. 2 show a comparison of our exact solution for cylindrical shells with experimental data[13]. Fig. 1 shows the frequency spectra for the supported shell, and in the Fig. 2 for the shell with a free edge. Also on Fig. 1 and Fig. 2 there are results calculated from the exact solution of the Soedel[14]. The agreement between present solution, experimental data and results of other researchers is obvious. A more complicated type of boundary conditions is presented in Table 2, i.e. one edge of the shell has elastic support, and the other edge is clamped. According to Eq.(11) when $k_u=k_v=k_w=k_\gamma=0$, we have clamped-free boundary conditions. Analysis of Table 2 shows that natural frequencies monotonically increase when the stiffness changes from 0 to 10e8.

**Figure 1.** Natural frequencies of simply supported cylindrical shell: $l=610\text{mm}$, $R=242.3\text{mm}$, $h=0.648\text{mm}$, $E=68,997\text{MPa}$, $\rho=2714.5\text{kg/m}^3$, $\mu=0.315$; experimental data[13] $(\odot) m=1, (\boxplus) m=2, (\bigcirc) m = 3$; exact solution; Soedel formula[14]

**Figure 2.** Natural frequencies of free edge cylindrical shell: $l=638\text{mm}$, $R=242.3\text{mm}$, $h=0.648\text{mm}$, $E=68,997\text{MPa}$, $\rho=2714.5\text{kg/m}^3$; experimental data[13] $(\odot) m=1, (\boxplus) m=2, (\bigcirc) m = 3$; exact solution; Soedel formula[14]
Table 2

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Conclusions. The accurate numerical solution of free vibration frequencies of cylindrical shell, based on the Donell-Mushtari theory, is obtained in explicit form. Eight main variables are selected, they are used to write out all the equations and boundary conditions. This formulation allowed us to solve a system of partial differential equations using series expansion. Also this formulation is suitable to address elastically supported edges, which are generalization of classical boundary conditions. The system of equations is solved accounting for deformation component and without, it influence is negligible. A comparison is made with the experimental results and with the data of other researchers, the correctness of present approach is obvious.

References
Список використаної літератури