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CONTACT INTERACTION OF A CIRCULAR PUNCH WITH A PRELIMINARILY STRESSED ISOTROPIC LAYER

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Using the linearized elasticity theory, we obtained the solution of the contact problem of pressure of a rigid circular punch of complex geometry on a preliminarily stressed isotropic layer. A numerical example of constructing the distribution function of contact stresses is considered. The effects of residual strains in the layer and the shape of a rigid punch on the distribution of contact stresses are analyzed.

The enhancement of the reliability and durability of structures and mechanisms is one of the most urgent tasks of modern building and mechanical engineering. As is known [2], the residual deformations are almost always present in elements of structures and articles of machines. The nature of their appearance can be very diverse: irreversible deformations (plasticity, creep), structural transformations in materials, change of the aggregate state in separate places of structures, mechanical, chemical, and technological processes, etc. The stresses arising in these cases, like any other ones, can cause fracture and accelerate certain phase transitions and corrosion. Consideration of residual deformations in the calculations of critical elements of structures, machines, and buildings allows one to estimate more exactly the safety factor of a material and, hence, to decrease essentially its consumption, by preserving the necessary functional characteristics of elements as a whole.

For this reason, studies of the contact interaction of elastic bodies with residual deformations are extremely crucial at the present time and will be such in the future.

Studies of problems of contact interaction of preliminarily stressed bodies in our country and abroad appeared in significant numbers only at the end of the last century. This is related, in the first turn, to the fact that linear elasticity theory does not consider the presence of residual stresses in bodies. In the general case, the strict statement of such problems requires the application of the apparatus of nonlinear elasticity theory. However, if the initial stresses are sufficiently high, we may restrict ourselves to its linearized version.

The present level of the linearized elasticity theory and mathematical methods together with the intense development of computers allow one to efficiently form various calculation models for a wide circle of problems. For example, the apparatus of the linearized elasticity theory was successfully used in works [5, 6] for the construction of a three-dimensional model of bounded elements and for studying the effects of the interaction of fibers during microdeformations in joints strengthened with isotropic and anisotropic fibers.

A fairly complete description and the classification of works devoted to the theory of contact interaction of preliminarily stressed bodies with rigid punches can be found in [1]. However, the question on the interaction of ring punches of complicated configurations with the elastic half-space or a layer with residual deformations remains insufficiently studied.

Let us consider the axisymmetric problem of the pressing of a rigid ring punch on a preliminarily stressed isotropic layer of thickness h that lies on a rigid absolutely smooth base.

The problem will be solved within the framework of the linearized elasticity theory with the use of the terminology and notation of [3]. We assume that the elastic potentials are continuous twice differentiable functions of algebraic invariants of the Green's tensor of deformations [3].

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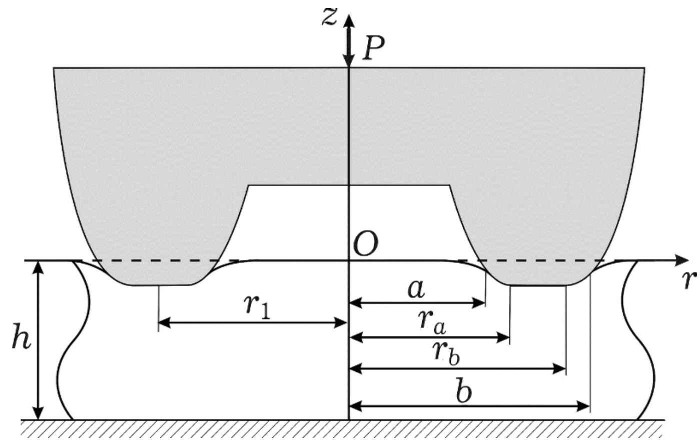

Fig. 1

We perform all calculations in the coordinates for a strained state y_i, which are connected with the Lagrange coordinates (which coincide with the Cartesian ones in the initial nonstrained state) by the relations $y_i = \lambda_i x_i$, $i = 1, 2, 3$, where λ_i are the coefficients of elongation of a linear element directed along the Cartesian axis x_i. We assume that the action of the punch causes small perturbations of the basic homogeneous stress-strain state in the layer. Let also the conditions

$$\lambda_1 = \lambda_2 \neq \lambda_3, \qquad S_0^{11} = S_0^{22} \neq S_0^{33} \tag{1}$$

be satisfied for this state, where S_0^{ii} are components of the tensor of initial stresses.

The punch presses on the layer progressively without rotation and friction under the action of a constant force P. It is formed by the rotation of two branches of parabolas around the common axis, which are conjugated at vertices by a segment of the line perpendicular to the axis of rotation. The axes of the parabolas forming the punch are parallel to the common axis of rotation coinciding with the line of action of the force P.

We chose the cylindrical coordinate system (O, r, θ, z) so that the coordinate plane (O, r, θ) coincides with the upper boundary plane of the layer, and the Oz axis coincides with the line of action of the force P (Fig. 1).

The boundary conditions of the problem posed read

$$\sigma_{rz}(r,0) = 0, \quad 0 \le r < \infty, \tag{2}$$

$$\sigma_{zz}(r,0) = 0, \quad 0 \le r \le a, \quad b \le r, \tag{3}$$

$$u_z(r,0) = w(r), \quad a \le r \le b, \tag{4}$$

$$\sigma_{rz}(r,-h) = 0, \quad 0 \le r < \infty, \tag{5}$$

$$u_z(r,-h) = 0, \quad 0 \le r < \infty. \tag{6}$$

The function $w(r)$ describing the shape of a rigid punch takes the form

$$w(r) = \begin{cases} w(a) + \frac{1}{2R_1}[(r_a - r)^2 - (r_a - a)^2], & a \le r < r_a, \\ w(a) - \frac{1}{2R_1}(r_a - a)^2, & r_a \le r < r_1, \\ w(b) - \frac{1}{2R_2}(r_b - b)^2, & r_1 \le r < r_b, \\ w(b) + \frac{1}{2R_2}[(r_b - r)^2 - (r_b - b)^2], & r_b \le r \le b, \end{cases} \tag{7}$$

where

$$r_1 = \frac{r_a + r_b}{2}$$

and R_1 and R_2 are the curvature radii of the parabolas forming the punch. Introducing two unknown functions $\varphi_1(r,z)$ and $\varphi_2(r,z)$, we can write the components of the vector of displacements and the tensor of contact stresses in the axisymmetric case in the form [2]

$$u_r = \frac{\partial}{\partial r}(\varphi_1 + \varphi_2) + z\frac{\partial^2}{\partial r \partial z}\varphi_2,$$

$$u_z = m_1\left[\left(\frac{\partial \varphi_1}{\partial z} + z\frac{\partial^2 \varphi_2}{\partial z^2}\right) + s_1\frac{\partial \varphi_2}{\partial z}\right],$$

$$\sigma_{zz} = c_{33}\left[\frac{\partial^2}{\partial z^2}(\varphi_1 + s\varphi_2) + z\frac{\partial^3 \varphi_2}{\partial z^3}\right],$$

$$\sigma_{rz} = c_{31}\left[\frac{\partial^2}{\partial r \partial z}(\varphi_1 + s_0\varphi_2) + z\frac{\partial^3 \varphi_2}{\partial r \partial z^2}\right]. \tag{8}$$

In this case, the functions $\varphi_1(r,z)$ and $\varphi_2(r,z)$ satisfy the equations

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + n_1\frac{\partial^2}{\partial z^2}\right)\varphi_i = 0, \quad i = 1,2. \tag{9}$$

Relations (8) are written in the general form within the theory of large (finite) deformations and for various versions of the theory of small initial deformations. They consider the presence of an elastic potential with arbitrary structure and are constructed for compressible and incompressible bodies without any limitations. The coefficients c_{31}, c_{33}, m_1, n_1, s, s_0, and s_1 in relations (8) and (9) depend on the character of the elastic potential and are determined for each specific case [3].

Applying the Hankel zero-order integral transformation to Eqs. (9),

$$\bar{\varphi}_i(\alpha,z) = \int_0^\infty r\varphi_i(r,\, z)J_0(\alpha r)\,dr\,,$$

we obtain the equation for the transforms

$$\left(n_1\frac{d^2}{dz^2}-\alpha^2\right)\bar{\varphi}_i(\alpha,z) = 0,\quad i=1,2\,, \tag{10}$$

whose solutions are chosen in the form

$$\bar{\varphi}_i(\alpha,z) = A_i(\alpha)\cosh(\alpha z)+B_i(\alpha)\sinh(\alpha z)\,. \tag{11}$$

Let us apply the inversion formula for the Hankel integral transformation to relations (11). Then relations (8) yield

$$u_r(r,z) = -\int_0^\infty \alpha^2\left\{A_1\cosh(\alpha z)+A_2\left[\cosh(\alpha z)+\alpha z\sinh(\alpha z)\right]\right.$$

$$\left.+\ B_1\sinh(\alpha z)+B_2\left[\sinh(\alpha z)+\alpha z\cosh(\alpha z)\right]\right\}J_1(\alpha r)\,d\alpha\,,$$

$$u_z(r,z) = m_1\int_0^\infty \alpha^2\left\{A_1\sinh(\alpha z)+A_2\left[s_1\sinh(\alpha z)+\alpha z\cosh(\alpha z)\right]\right.$$

$$\left.+\ B_1\cosh(\alpha z)+B_2\left[s_1\cosh(\alpha z)+\alpha z\sinh(\alpha z)\right]\right\}J_0(\alpha r)\,d\alpha\,,$$

$$\sigma_{rz}(r,z) = -c_{31}\int_0^\infty \alpha^3\left\{A_1\sinh(\alpha z)+A_2\left[s_0\sinh(\alpha z)+\alpha z\cosh(\alpha z)\right]\right.$$

$$\left.+\ B_1\cosh(\alpha z)+B_2\left[s_0\cosh(\alpha z)+\alpha z\sinh(\alpha z)\right]\right\}J_1(\alpha r)\,d\alpha\,,$$

$$\sigma_{zz}(r,z) = c_{33}\int_0^\infty \alpha^3\left\{A_1\cosh(\alpha z)+A_2\left[s\cosh(\alpha z)+\alpha z\sinh(\alpha z)\right]\right.$$

$$\left.+\ B_1\sinh(\alpha z)+B_2\left[s\sinh(\alpha z)+\alpha z\cosh(\alpha z)\right]\right\}J_0(\alpha r)\,d\alpha\,. \tag{12}$$

On the upper boundary plane of the layer (for $z=0$) with regard for relations (12), we have

$$\sigma_{zz} = c_{33}\int_0^\infty \alpha^3 (A_1 + A_2\, s) J_0(\alpha r)\, d\alpha\,, \tag{13}$$

$$\sigma_{rz} = -c_{31}\int_0^\infty \alpha^3 (B_1 + B_2\, s_0) J_1(\alpha r)\, d\alpha\,, \tag{14}$$

$$u_z = m_1\int_0^\infty \alpha^2 (B_1 + B_2\, s_1) J_0(\alpha r)\, d\alpha\,. \tag{15}$$

On the lower boundary plane of the layer (for $z = -h$), relations (12) yield

$$\sigma_{rz} = -c_{31}\int_0^\infty \alpha^3 \{-A_1 \sinh(\alpha h) + A_2 [-s_0 \sinh(\alpha h) - \alpha h \cosh(\alpha h)]$$

$$+ B_1 \cosh(\alpha h) + B_2 [s_0 \cosh(\alpha h) + \alpha h \sinh(\alpha h)]\} J_1(\alpha r)\, d\alpha\,,$$

$$u_z = m_1\int_0^\infty \alpha^2 \{-A_1 \sinh(\alpha h) + A_2 [-s_1 \sinh(\alpha h) - \alpha h \cosh(\alpha h)]$$

$$+ B_1 \cosh(\alpha h) + B_2 [s_1 \cosh(\alpha h) + \alpha h \sinh(\alpha h)]\} J_0(\alpha r)\, d\alpha\,. \tag{16}$$

Let the boundary condition (2) be satisfied. Then we obtain the following relationship between the functions B_1 and B_2 from equality (14):

$$B_1 + B_2\, s_0 = 0 \quad \Rightarrow \quad B_1 = -s_0\, B_2\,. \tag{17}$$

Substituting (17) in relation (16) and taking the boundary conditions (5) and (6) into account, we obtain the system for the unknowns A_1 and A_2:

$$A_1 \sinh(\alpha h) + A_2 [s_0 \sinh(\alpha h) + \alpha h \cosh(\alpha h)] = B_2 \alpha h \sinh(\alpha h)\,,$$

$$A_1 \sinh(\alpha h) + A_2 [s_1 \sinh(\alpha h) + \alpha h \cosh(\alpha h)] = B_2 [(s_1 - s_0) \cosh(\alpha h) + \alpha h \sinh(\alpha h)]. \tag{18}$$

Solving (18), we obtain the following formulas for A_1 and A_2 in terms of the function B_2:

$$A_1 = -\frac{\alpha h + s_0 \sinh(\alpha h) \cosh(\alpha h)}{\sinh^2(\alpha h)} B_2\,,$$

$$A_2 = \frac{\sinh(\alpha h) \cosh(\alpha h)}{\sinh^2(\alpha h)} B_2\,. \tag{19}$$

With regard for relations (17) and (19), formulas (13) and (15) take the form

$$\sigma_{zz} = c_{33}\int_0^\infty \alpha^3 \frac{(s-s_0)\sinh(\alpha h)\cosh(\alpha h)-\alpha h}{\sinh^2(\alpha h)} B_2 J_0(\alpha r)\,d\alpha\,, \tag{20}$$

$$u_z = m_1(s_1-s_0)\int_0^\infty \alpha^2 B_2 J_0(\alpha r)\,d\alpha\,. \tag{21}$$

Let the boundary condition (3) be satisfied. In view of (20), we have

$$c_{33}\int_0^\infty \alpha^3 \frac{(s-s_0)\sinh(\alpha h)\cosh(\alpha h)-\alpha h}{\sinh^2(\alpha h)} B_2 J_0(\alpha r)\,d\alpha = 0\,, \quad 0 \le r \le a, \quad b \le r\,. \tag{22}$$

We now introduce the unknown function $x(r)$, $a \le r \le b$. Using it, we extend relation (22) onto the interval $0 \le r < \infty$,

$$c_{33}\int_0^\infty \alpha^3 \frac{(s-s_0)\sinh(\alpha h)\cosh(\alpha h)-\alpha h}{\sinh^2(\alpha h)} B_2 J_0(\alpha r)\,d\alpha = x(r)\{U(r-a)-U(r-b)\}, \quad 0 \le r < \infty, \tag{23}$$

where $U(r)$ is the Heaviside function.

The function $x(r)$ determine the distribution of contact stresses under the punch. In view of its continuity and zero value on the contact domain boundary (for $r = a$ and $r = b$), we represent $x(r)$ in the form of a part of the generalized Fourier series in the functions

$$L_n(r) = J_0\left(\frac{\gamma_n}{a}r\right)Y_0(\gamma_n) - Y_0\left(\frac{\gamma_n}{a}r\right)J_0(\gamma_n),$$

i.e., in the form

$$x(r) = \sigma_{zz}(r, 0) = \sum_{n=1}^{N} a_n L_n(r), \tag{24}$$

where a_n are the unknown coefficients, and γ_n are positive roots of the equation

$$J_0\left(\frac{b}{a}t\right)Y_0(t) - Y_0\left(\frac{b}{a}t\right)J_0(t) = 0\,.$$

Applying the inversion formula for the Hankel integral transformation to relation (23) and taking representation (24) into account, we obtain

$$\alpha^2 B_2 \frac{(s-s_0)\sinh(\alpha h)\cosh(\alpha h)-\alpha h}{\sinh^2(\alpha h)} = \frac{1}{c_{33}}\int_a^b r\sum_{n=1}^N a_n L_n(r) J_0(\alpha r)\,dr$$

$$= \frac{1}{c_{33}}\sum_{n=1}^N a_n \int_a^b r\,L_n(r) J_0(\alpha r)\,dr. \qquad (25)$$

Let us introduce the notation

$$\Phi_n(\alpha) = \int_a^b r\,L_n(r) J_0(\alpha r)\,dr = \frac{\gamma_n a^2}{\gamma_n^2-(\alpha a)^2}\left\{\frac{b}{a}\left[J_1\left(\frac{b}{a}\gamma_n\right)Y_0(\gamma_n)-Y_1\left(\frac{b}{a}\gamma_n\right)J_0(\gamma_n)\right]J_0(\alpha b)\right.$$

$$\left.-\left[J_1(\gamma_n)Y_0(\gamma_n)-Y_1(\gamma_n)J_0(\gamma_n)\right]J_0(\alpha a)\right\}.$$

Then relations (25) yield

$$\alpha^2 B_2 = \frac{\sinh^2(\alpha h)}{(s-s_0)\sinh(\alpha h)\cosh(\alpha h)-\alpha h}\frac{1}{c_{33}}\sum_{n=1}^N a_n\,\Phi_n(\alpha). \qquad (26)$$

Substituting relations (26) in equality (21), we obtain

$$u_z(r) = m_1(s_1-s_0)\int_0^\infty \alpha^2 B_2 J_0(\alpha r)\,d\alpha$$

$$= \frac{m_1(s_1-s_0)}{c_{33}(1+m_1)\ell_1 n_1}\int_0^\infty \frac{\sinh^2(\alpha h)}{(s-s_0)\sinh(\alpha h)\cosh(\alpha h)-\alpha h}\sum_{n=1}^N a_n \Phi_n(\alpha) J_0(\alpha r)\,d\alpha$$

or

$$u_z(r) = -\omega\sum_{n=1}^N a_n\int_0^\infty \Delta(\alpha)\Phi_n(\alpha) J_0(\alpha r)\,d\alpha. \qquad (27)$$

Here, we denote

$$\omega = \frac{(s_0-s_1)m_1}{c_{33}(1+m_1)\ell_1 n_1},$$

$$\Delta(\alpha) = \frac{\sinh^2(\alpha h)}{(s-s_0)\sinh(\alpha h)\cosh(\alpha h)-\alpha h} = \frac{1-2e^{-2\alpha h}+e^{-4\alpha h}}{(s-s_0)\left[1-e^{-4\alpha h}\right]-4\alpha h e^{-2\alpha h}}.$$

Let the boundary condition (4) be satisfied. Substituting the formulas for $w(a)$ and $w(b)$ following from (27) in relation (7), we obtain

$$-\omega \sum_{n=1}^{N} a_n \int_0^\infty \Delta(\alpha)\Phi_n(\alpha)\{J_0(\alpha r) - J_0(\alpha a)\} d\alpha = w_1^*(r),$$

$$-\omega \sum_{n=1}^{N} a_n \int_0^\infty \Delta(\alpha)\Phi_n(\alpha)\{J_0(\alpha r) - J_0(\alpha b)\} d\alpha = w_2^*(r). \tag{28}$$

Here, we use the notation

$$w_1^*(r) = \begin{cases} -\dfrac{1}{2R_1}\left[(r_a - a)^2 - (r_a - r)^2\right], & a \le r < r_a, \\ -\dfrac{1}{2R_1}(r_a - a)^2, & r_a \le r < r_1, \end{cases}$$

$$w_2^*(r) = \begin{cases} -\dfrac{1}{2R_2}(r_b - b)^2, & r_1 \le r < r_b, \\ -\dfrac{1}{2R_2}\left[(r_b - b)^2 - (r_b - r)^2\right], & r_b \le r \le b. \end{cases}$$

Multiplying relations (28) by $r L_q(r)$ and integrating the formulas obtained over r from a to b, we obtain

$$\omega \sum_{n=1}^{N} a_n \int_0^\infty \Delta(\alpha)\Phi_n(\alpha)\{\Phi_q(\alpha) - \mathcal{R}_q^{(1)} J_0(\alpha a) - \mathcal{R}_q^{(2)} J_0(\alpha b)\} d\alpha$$

$$= \frac{1}{2R_1}\left\{(r_a - a)^2 \mathcal{R}_q^{(1)} - \int_a^{r_a} (r_a - r)^2 r L_q(r)\, dr\right\}$$

$$+ \frac{1}{2R_2}\left\{(r_b - b)^2 R_q^{(2)} - \int_{r_b}^{b} (r_b - r)^2 r L_q(r)\, dr\right\}, \quad q = 1, \dots, N, \tag{29}$$

where

$$\mathcal{R}_q^{(1)} = \int_a^{r_1} r L_q(r)\, dr, \qquad \mathcal{R}_q^{(2)} = \int_{r_1}^{b} r L_q(r)\, dr.$$

Using the method of superposition and introducing the notation

$$a_n = \frac{1}{\omega}\left[\frac{1}{2R_1}a_n^{(1)} + \frac{1}{2R_2}a_n^{(2)}\right], \qquad z_1 = \frac{1}{2R_1}, \qquad z_2 = \frac{1}{2R_2}, \tag{30}$$

we deduce from (29) the following two systems for the unknowns $a_n^{(1)}$ and $a_n^{(2)}$:

$$\sum_{n=1}^{N} a_n^{(1)} K_q = (r_a - a)^2 \mathcal{R}_q^{(1)} - \int_a^{r_a} (r_a - r)^2 r L_q(r)\, dr,$$

$$\sum_{n=1}^{N} a_n^{(2)} K_q = (r_b - b)^2 \mathcal{R}_q^{(2)} - \int_{r_b}^{b} (r_b - r)^2 r L_q(r)\, dr, \quad q = 1, \ldots, N.$$

Here,

$$K_q = \int_0^{\infty} \Delta(\alpha)\Phi_n(\alpha)\left\{\Phi_n(\alpha) - \mathcal{R}_q^{(1)} J_0(\alpha a) - \mathcal{R}_q^{(2)} J_0(\alpha b)\right\} d\alpha.$$

The quantities z_i in relations (30) can be determined from the condition of equilibrium of the punch

$$2\pi \int_a^b r\sigma_{zz}(r, 0)\, dr = -P \tag{31}$$

and from the equality of vertical displacements of the upper boundary plane of the layer for $r = r_a$ and $r = r_b$:

$$u_z(r_a) = u_z(r_b). \tag{32}$$

Condition (31) and relations (30) and (24) yield

$$\int_a^b r\, x(r)\, dr = \sum_{n=1}^{N} a_n \int_a^b r L_n(r)\, dr = \frac{1}{\omega}\left\{z_1 \sum_{n=1}^{N} a_n^{(1)} \mathcal{R}^q + z_2 \sum_{n=1}^{N} a_n^{(2)} \mathcal{R}^q\right\}, \tag{33}$$

$$z_1 \sum_{n=1}^{N} a_n^{(1)} \mathcal{R}^q + z_2 \sum_{n=1}^{N} a_n^{(2)} \mathcal{R}^q = -\frac{P}{2\pi}\omega,$$

where

$$\mathcal{R}^q = \int_a^b r L_q(r)\, dr.$$

From condition (32) and relations (7) and (27), we obtain

$$w(a) - \frac{1}{2R_1}(r_a - a)^2 = w(b) - \frac{1}{2R_2}(r_b - b)^2,$$

$$z_1(r_a - a)^2 - z_2(r_b - b)^2 = w(a) - w(b)$$

$$= \omega \sum_{n=1}^{N} a_n \int_0^\infty \Delta(\alpha)\Phi_n(\alpha)\{J_0(\alpha a) - J_0(\alpha b)\}d\alpha$$

$$= \omega \sum_{n=1}^{N} a_n M_n = \sum_{n=1}^{N} \{z_1 a_n^{(1)} + z_2 a_n^{(2)}\} M_n$$

$$= z_1 \sum_{n=1}^{N} a_n^{(1)} M_n + z_2 \sum_{n=1}^{N} a_n^{(2)} M_n,$$

where

$$M_n = \int_0^\infty \Delta(\alpha)\Phi_n(\alpha)\{J_0(\alpha b) - J_0(\alpha a)\}d\alpha.$$

Finally, we obtain the following equations for the unknowns z_1 and z_2:

$$z_1\left\{(r_a - a)^2 - \sum_{n=1}^{N} a_n^{(1)} M_n\right\} + z_2\left\{-(r_b - b)^2 - \sum_{n=1}^{N} a_n^{(2)} M_n\right\} = 0. \tag{34}$$

Let us make the change

$$z_i^* = z_i \omega \frac{2\pi}{P}, \quad i = 1,2, \quad \Rightarrow \quad z_i = \omega \frac{P}{2\pi} z_i^* \tag{35}$$

in Eqs. (33) and (34). Then we obtain a system of two equations for the unknowns z_1^* and z_2^*:

$$z_1^* \sum_{n=1}^{N} a_n^{(1)} \mathcal{R}^q + z_2^* \sum_{n=1}^{N} a_n^{(2)} \mathcal{R}^q = -1,$$

$$z_1^*\left\{(r_a - a)^2 - \sum_{n=1}^{N} a_n^{(1)} M_n\right\} + z_2^*\left\{-(r_b - b)^2 - \sum_{n=1}^{N} a_n^{(2)} M_n\right\} = 0. \tag{36}$$

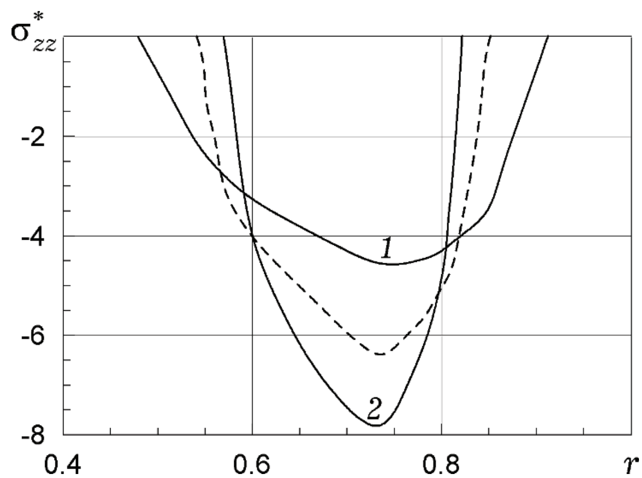

Fig. 2

Solving system (36) and taking relations (24), (30), and (35) into account, we obtain the formula for the distribution of contact stresses under the punch:

$$\sigma_{zz}(r,0) = \frac{P}{2\pi}\left\{ z_1^* \sum_{n=1}^{N} a_n^{(1)} L_n(r) + z_2^* \sum_{n=1}^{N} a_n^{(2)} L_n(r) \right\}.$$

With the help of the solved problem, we now study the influence of the shape of a rigid punch and the presence of residual deformations in the layer on the character of the distribution of contact stresses. We carry out the numerical analysis in two cases:

– the layer is a compressible body with an elastic potential of the harmonic type (Figs. 2 and 4);

– the layer is a noncompressible body with the Barten’ev–Khazanovich potential (Figs. 3 and 5).

In Figs. 2 and 3, we present graphs of the dimensionless function

$$\sigma_{zz}^* = \frac{2\pi}{P}\sigma_{zz}(r,0)$$

in the case where the rectilinear section in the base of the punch is absent, and the configuration of the punch is determined by the following values of the parameters:

$$r_a = r_b = 0.7, \qquad R_1 = \frac{\pi}{14}\frac{1}{\omega P}, \qquad R_2 = \frac{\pi}{24}\frac{1}{\omega P}.$$

The dotted lines in the figures correspond to the absence of residual deformations in the layer $(\lambda_1 = 1)$. Curves 1 and 2 are related to the presence of compressive $(\lambda_1 < 1)$ and tensile $(\lambda_1 > 1)$ residual deformations, respectively. The performed numerical analysis allows us to assert that the appearance of tensile residual deformations in a body causes the narrowing of the contact region and an increase of the moduli of contact stresses. The appearance of compressive deformations induces the extension of the contact region and a decrease of the moduli of contact stresses.

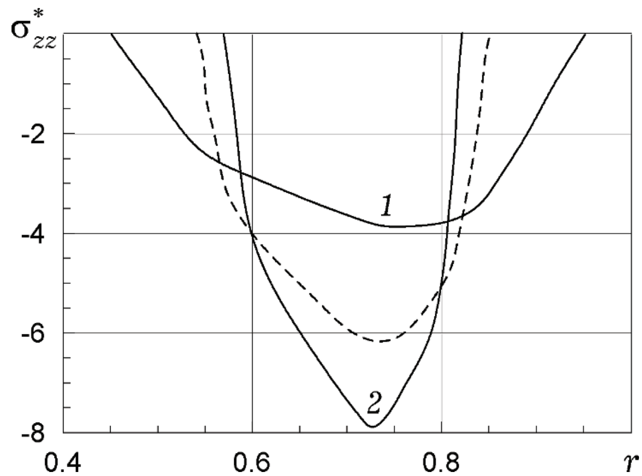

Fig. 3

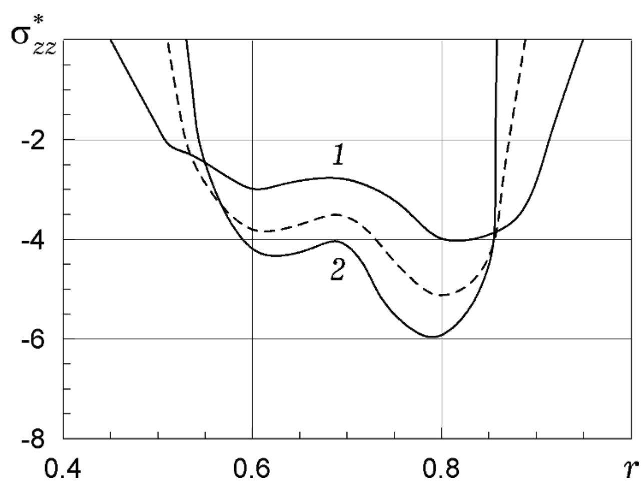

Fig. 4

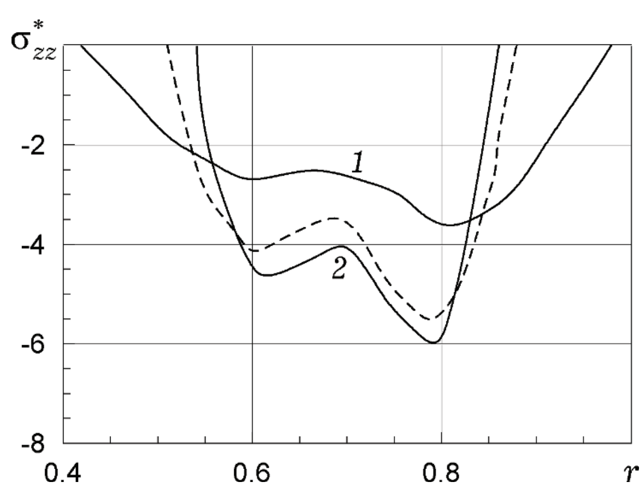

Fig. 5

In Figs. 4 and 5, we give graphs of the function σ^*_{zz} for the following values of the parameters:

$$r_a = 0.65\,, \qquad r_b = 0.75\,, \qquad R_1 = \frac{\pi}{14}\frac{1}{\omega P}\,, \qquad R_2 = \frac{\pi}{24}\frac{1}{\omega P}\,.$$

The analysis of the results obtained allows us to conclude that the appearance of the rectilinear section $\left[r_a, r_b\right]$ in the base of the punch causes a decrease in the modulus of contact stresses and a displacement of the extremum points to the contact region boundary. The reliability of the conclusions is supported by their agreement with the results obtained, e.g., in [4].

REFERENCES

1. S. Yu. Babich, A. N. Guz, and V. B. Rudnitskii, "Contact problems for prestressed elastic bodies and rigid and elastic punches," *Int. Appl. Mech.*, **40**, No. 7, 744–765 (2004).
2. A. N. Guz and V. B. Rudnitskii, *Foundations of the Theory of Contact Interaction of Elastic Bodies with Initial (Residual) Stresses* [in Russian], Mel'nik, Khmel'nitskii (2006).
3. A. N. Guz, *Mechanics of Brittle Fracture of Materials with Initial Stresses* [in Russian], Naukova Dumka, Kiev (1983).
4. O. M. Guz and V. B. Rudnyts'kyi, "Contact interaction of bodies with initial (residual) stresses," in: *Problems of Simulation of Modern Technologies: Collection of Scientific Reports of the Intern. Sci.-Techn. Conference* [in Ukrainian], Khmel'nyts'kyi Univ., Khmel'nyts'kyi (2004), pp. 5–35.
5. J. Harich, Y. Lapusta, and W. Wagner, "3D FE-modeling of surface and anisotropy effects during micro-buckling in fiber composites," *Compos. Struct.*, **89**, No. 4, 551–555 (2009).
6. Y. Lapusta, J. Harich, and W. Wagner, "Three-dimensional FE model for fiber interaction effects during micro-buckling in composites with isotropic and anisotropic fibers," *Commun. Numer. Methods Eng.*, **24**, No. 12, 2206–2215 (2008).