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SEQUENCES OF SEMIGROPS OF NONLINEAR OPERATORS AND THEIR APPLICATIONS TO STUDY THE CAUCHY PROBLEM FOR PARABOLIC EQUATIONS

Mikola Yaremenko

National Technical University of Ukraine «Igor Sikorsky Kyiv Polytechnic Institute»

Summary. We consider the operator function of exponential type, studied the link between these functions (semigroup) and Cauchy problem for differential parabolic equation. We establish conditions under which the semigroup is associated with Cauchy problem; we investigate semigroups sequences and their convergence to function of exponential type which is semigroup. We consider maximal dissipative operators and maximum semigroups. We study the problem of existence of the solution of nonlinear partial differential equations of parabolic type with measurable coefficients, nonlinear term which satisfies the forms – bounded conditions.

Key words: quasi-linear differential equations, dissipative operators, the method of forms, semigroup, maximal operators, sequence of semigroups.

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Introduction. Consider the Cauchy problem for a parabolic equation in the form
\[
\frac{du}{dt}(t) \in A(u(t)), \quad u(t) \in L^p(R^l, d^l x),
\]
for almost all \( t \in [0, t_0] \), with the initial condition \( u(0) = u_0 \), \( u_0 \in D(A) \), where the operator \( A : L^p(R^l, d^l x) \rightarrow L^p(R^l, d^l x) \) is determined by the operator \( A^p : W^p \rightarrow W^p \), which operates as follows: \( h^p(u, v) = \langle A^p(u), v \rangle \). The form \( h^p(u, v) \) is based on the left part of elliptic equation
\[
\lambda u - \sum_{i,j=1}^l \frac{\partial}{\partial x_i} \left( a_{ij}(x, u) \frac{\partial u}{\partial x_j} \right) + b(x, u, \nabla u) = f,
\]
where \( u(x) \) is the unknown function, \( \lambda > 0 \) – a real number, and \( f(x) \) – given function \([8, 9]\). Here \( b(x, u, \nabla u) \) – a function of three variables: the dimension vector \( l \), scalar, dimension vector \( l \). Dimensional matrix \( a_{ij}(x, u) \) of dimension \( l \times l \)
\[
v(u) \sum_{i=1}^l \xi_i^2 \leq \sum_{i<j} a_{ij}(x, u) \xi_i \xi_j \leq \mu(u) \sum_{i=1}^l \xi_i^2 \quad \forall \xi \in R^l,
\]
satisfies ellipticity condition: for almost all \( x \in R^l \) \([4-5, 8, 9]\).

Let us construct form \( h^p : W^p \times W^p \rightarrow R \), \( h^p(u, v) = \lambda \langle u, v \rangle + \langle \nabla v \circ a \circ \nabla u \rangle + \langle b(x, u, \nabla u), v \rangle \), which is assumed to be specified for all elements \( u \in W^p(R^l, d^l x), v \in W^p(R^l, d^l x) \).

The function \( b(x, u, \nabla u) \) is a measurable function of its arguments and \( b \in L^1_{loc}(R^l) \); function almost everywhere satisfies
\[
|b(x, u, \nabla u)| \leq \mu_1(x)|\nabla u| + \mu_2(x)|b| + \mu_3(x),
\]
We introduce the class of functions
\[
PK_{\beta}(A) = \{ f \in L^1_{loc}(R^l, d^l x) : \langle h f, h \rangle \leq \beta \langle \nabla h \circ a \circ \nabla h \rangle + c(\beta) \| h \|_{L^2}^2 \},
\]
where \( \beta > 0 \), \( c(\beta) \in R^l \). Features
\[ \mu^2 \in PK_\beta(A), \ \mu_4 \in PK_\beta(A). \] Growth of the function \( b(x,y,z) \) almost everywhere satisfies the condition:
\[ |b(x,u,\nabla u) - b(x,v,\nabla v)| \leq \mu_4(x)|\nabla(u-v)| + \mu_4(x)|u-v| \] where \( \mu^2 \in PK_\beta(A), \ \mu_4 \in PK_\beta(A) \) \[8, 9\].

**Preliminary information.**

**Definition 1.** A set of one-parameter nonlinear operators \( T_t, t \geq 0 \), is called a continuous one-parameter semigroup if the following conditions are met: for any fixed \( t \geq 0 \) operator \( T_t \) is a continuous nonlinear operator which operates from \( L^p(\mathbb{R}^i,d^i x) \) in \( L^p(\mathbb{R}^i,d^i x) \); for any fixed \( f \in L^p(\mathbb{R}^i,d^i x) \) set of elements \( T_t f \) is strongly continuous on \( t \); there exists a property of group \( T_{s+t} = T_s T_t \), for \( t, s \geq 0 \) at \( T_0 = I \), where 1 - the identical motion.

**Definition 2.** Let function \( u_n(t) \in D(A) \subseteq C_{L^p(\mathbb{R}^i,d^i x)}[0,t], \ n \in N, \) satisfy (in the classical sense) equation \[ \frac{d}{dt} u_n(t) = A_n u_n(t), \ n \in N, \] where \( A_n, n \in N, \ - \) motion of \( u-\nu n^{-1} \rightarrow v \) at \( u \in D(A), \ v \in Au \). If the sequence \( \{u_n(t), n \in N\} \) matches evenly with \( u(t) \in D(A), \ t \in [0,t_o] \) in the strong topology, and there is a subsequence \( \{\frac{d}{dt} u_{n_k}(t), n_k \in N\} \) that matches a \( \sigma \)-weak topology to the element \( \frac{d}{dt} u(t) \in L_{L^p(\mathbb{R}^i,d^i x)}[0,t], \) the element \( u(t) \in D(A) \) is called a solution of generalized parabolic equation.

We know that if \( u(t) \) is an absolutely continuous function than \( u(t) \) is differentiable for almost all \( t \) because \( L^p(\mathbb{R}^i,d^i x) \) is a reflexive Banach space, and can be written by the integral of its derivative, which exists for almost all \( t \).

**Definition 3.** Own solution of the Cauchy problem for a parabolic equation is a function \( u(t), \) if \( u(t) \in D(A) \) and this function is absolutely continuous for almost all \( t \) and satisfies for almost all \( t \) this generalized parabolic equation.

Denote with \( C_{L^p(\mathbb{R}^i,d^i x)}[0,t], \) the space of all \( L^p(\mathbb{R}^i,d^i x) \) - significant highly continuous functions on the interval of real axis \( [0,t], \) i.e. if \( u(\cdot) \in C_{L^p(\mathbb{R}^i,d^i x)}[0,t], \) then \( u:[0,t] \rightarrow L^p(\mathbb{R}^i,d^i x) \); and through \( L_t = L_{L^p(\mathbb{R}^i,d^i x)}[0,t], \) - space of all \( L^p(\mathbb{R}^i,d^i x) \) - significant highly integrable functions on the interval \( [0,t], \) that is, if \( u(\cdot) \in L_{L^p(\mathbb{R}^i,d^i x)}[0,t], \) then \( u:[0,t] \rightarrow L^p(\mathbb{R}^i,d^i x) \) and
\[ \|u\| = \int_0^t \|u(s)\| ds < \infty. \]

We assume that the operator \( A \) is valid from \( L^p(\mathbb{R}^i,d^i x) \) to \( L^p(\mathbb{R}^i,d^i x) \) and is one that generates mapping from \( C_{L^p(\mathbb{R}^i,d^i x)}[0,t] \) to \( L_{L^p(\mathbb{R}^i,d^i x)}[0,t] \) which can be determined by the rule \( C_{L^p(\mathbb{R}^i,d^i x)}[0,t] \ni u \rightarrow \{v \in L_{L^p(\mathbb{R}^i,d^i x)}[0,t]: v(s) \in Au(s) \text{ almost everywhere } \{s\}\}, \) this mapping is also denoted by the letter \( A. \)
Nonlinear semigroups and local generators.

Definition 4. Semigroup $T_t$ is called the maximum compression semigroup if there is no compression semigroup with a broader definition domain to which it may be extended.

Remark. Any compression semigroup can be extended to a maximum compression semigroup.

Remark. Maximum dissipative operator does not necessarily generate a maximum compression semi-group.

Lemma 1. Let $\{B_\alpha : \alpha \in \Gamma\}$ and $\{B'_\alpha : \alpha \in \Gamma\}$ — two systems of areas in $L^p(R^t, d^tx)$.
$$B_\alpha = \{f \in L^p(R^t, d^tx) : \|f - f_\alpha\| \leq r_\alpha\}, \quad B'_\alpha = \{f \in L^p(R^t, d^tx) : \|f - f'_\alpha\| \leq r'_\alpha\}.$$ If
$$\|f_\alpha - f_\beta\| \leq \|f'_\alpha - f'_\beta\| \quad \text{and} \quad r_\alpha \leq r'_\alpha \quad \forall \alpha, \beta \in \Gamma,$$ then because of $\bigcap_{\alpha \in \Gamma} B_\alpha \neq \phi$ stems condition
$$\bigcap_{\alpha \in \Gamma} B'_\alpha \neq \phi.$$ .

Lemma 2. Let $\Omega$ — convex closed shell $D(T_t)$. For any fixed natural number $k$ there is mapping $U_k : \Omega \to \Omega$ that $\|U_k f - U_k q\| \leq \|f - q\| \quad \forall f, q \in \Omega$ and $U_k f = T_{2^k} f \quad \forall f \in D(T_t)$.

If $f_0$ and $f'_0$ of $\Omega$ satisfy correlation $\|f_0^* - f\| \leq \|f'_0 - f\| \quad \forall f \in D(T_t)$, there is an $U_k$ expansion of $T_{2^k}$ that $U_k f_0' = f'_0$.

Proof. Let us assume that set $\Omega - D(T_t)$ is ordered like $\{f_\alpha\}$. Using transfinite induction, we construct the map $U_k$. Suppose that $U_k$ is a reflection of compression that is defined for
$$\forall f_\beta : \beta < \alpha \forall q \in D(T_t) : U_k q = T_{2^k} q.$$ Let $B(f - \varphi_\alpha; \varphi_\beta) = \{f_1 : \| f_1 - \varphi_\beta \| \leq \| f - \varphi_\alpha \|\}$.

Thus, for systems of areas $\{B(f_\alpha - f_\beta; f'_\beta), B(f_\alpha - q; q) : \beta < \alpha, q \in D(T_t)\}$ and $\{B(f_\alpha - f_\beta; U_k f_\beta), B(f_\alpha - q; U_k q) : \beta < \alpha, q \in D(T_t)\}$, then
$$\bigcap_{\beta < \alpha} B(f_\alpha - f_\beta; f'_\beta) \cap_{\beta \in D(T_t)} B(f_\alpha - q; U_k q) \neq \phi,$$ as $f_\alpha$ belongs to this intersection, it follows:
$$\bigcap_{\beta < \alpha} B(f_\alpha - f_\beta; U_k f_\beta) \cap_{\beta \in D(T_t)} B(f_\alpha - q; U_k q) \leq B_\alpha^0 \neq \phi.$$ .

Denote the projection $L^p(R^t, d^tx) \to \Omega$ with $p$, thus $pf = q$ at
$$\inf_{q \in \Omega} \|q_1 - f\|.$$
As inequality for the norms is true:
$$\|p\phi - f\| \leq \|\phi - f\| \forall f \in \Omega, \quad \phi \in L^p(R^t, d^tx)$$
therefore, there is identity $\varphi_\beta \in B_\alpha^0$, from where we get $pf_\beta \in B_\alpha^0$.

Select an item $f_\alpha^0 \in B_\alpha^0 \cap \Omega$ and let $U_k f_\alpha = f_\alpha^0$, then $U_k$ compression is on $D(T_t) \cup \{f_\beta : \beta \leq \alpha\}$.

Using transfinite induction we have received the necessary map $U_k$. The Lemma statement is proved.

Consider the semi group
$$T^a = \left\{T^a_t : t = \frac{j}{2^n}, \quad j = 0, 1, 2, \ldots \right\},$$ where $T^a_{2^k} = U_k^a$ and
$$T^a_{t+s} = T^a_t T^a_s \quad \text{when} \quad t = \frac{j}{2^k}, \quad s = \frac{j}{2^k}, \quad \text{and the set of maps} \{-U^a_\alpha : \alpha \in \Gamma\} \text{from Lemma 2}.$$ .

Denote with $\tau_k$ the set $\{T^a : \alpha\}$ and define canonical map for $n \geq k$ $J_{nk} : \tau_n \to \tau_k$ as
sequences of semigroups of nonlinear operators and their applications to study the cauchy problem for parabolic equations

\[ J_{n \cdot k} T^a = T^b \]

at

\[ T_{2 \cdot k}^a = T_{2 \cdot k}^b, \]

where \( T^a \in \tau_k, T^b \in \tau_n \). Noting that \( J_{m \cdot n} : J_{n \cdot k} : \tau_m \rightarrow \tau_n \rightarrow \tau_k \), \( J_{m \cdot n} J_{n \cdot k} T^a = J_{m \cdot n} J_{n \cdot k} T^b = T^\gamma \), \( T_{2 \cdot k}^a = T_{2 \cdot k}^b \), \( T^a \in \tau_k, T^b \in \tau_n, T^\gamma \in \tau_m \), obtain approval

\[ J_{m \cdot n} J_{n \cdot k} = J_{m \cdot k}, D(J_{n \cdot k}) = \tau_n. \]

Therefore, we can put \( A_k^a = (T_{2 \cdot k} - I) \) for \( T^a \in \tau_k \).

**Theorem 1.** Scope definition maximum compression semigroup \( \{ T_t \} \) is a closed convex set that is not contained in any closed hyperplane.

**Proof.** On the opposite, let \( D(T_t) \) be its own subset of convex closed shell \( \Omega \).

Define dissipative operator \( A \) tightly defined in \( \Omega \) as \( Aq = \{ 2^n (q - f) : \exists \eta, q_n(f) = q \} \), where

\[ q_n(f) = \lim_{x \to \infty} (1 - 2^n A^a_{\infty})^{-1} f_b - f_0, \]

where \( f_b \) is part of \( q_0 - Aq_0 \), and \( A_{\infty} \) is a mapping of: \( f - 2^n f_i \rightarrow f_1, f \in D(A), f_1 \in AF \).

Let \( P : L^p(R^d, d^t x) \rightarrow \Omega \) be the projection, thus \( Pf = q \in \Omega \subset L^p(R^d, d^t x) \) and \( \| q - f \| = \inf_{q \in \Omega} \| q - f \| \). Define the sequence \( u_n^{m}(t) \) by placing the induction

\[ u_n^{m+1}(t) = P(q_0 + \int_0^t A_n u_n^m(s) ds) \]

Then

\[ u_n^{m+1}(t) = \Omega \subset L^p(R^d, d^t x), \quad \text{and} \]

\[ \| u_n^{m+1}(t) - u_n^m(t) \| = \left\| \int_0^t A_n u_n^m(s) ds \right\| \leq \left\| \int_0^t A_n u_n^m(s) - A_n u_n^{m+1}(s) \right\| ds \]

As a result of Lipchitz condition \( A_n \) we have \( \sum_{m \rightarrow \infty} \| u_n^{m+1}(t) - u_n^m(t) \| < \infty \), so there

\[ u_n(t) = \lim_{m \rightarrow \infty} u_n^m(t) \]

Resulting from theorem given earlier, \( u_n(t) \) satisfies the equation in the approximate Cauchy problem and \( \{ u_n(t) \}_{n=0}^{\infty} \) tends to function \( u_n(t) \) evenly at \( t \) on \( [0, t_0] \) and function \( u_n(t) \) satisfies the equation in the initial Cauchy problem.
Show that 

\[ \tilde{T}_i(f) = \begin{cases} T_i f, & n p u f \in D(T_i), \\ u(t+s), & n p u f = u(s), s \geq 0, \end{cases} \]

is really a compression semigroup.

Because of dissipativity of \( A \), we get \( \| u(s+t) - u(s_t + t) \| \leq \| u(s) - u(s_t) \| \) at \( s, s_t, t \geq 0 \) thus \( \| u(s+t) - T_i f \| \leq \| u(s) - f \| \) at \( f \notin D(T_i) \), \( s, t > 0 \).

Let \( s, t, s + t \in (0, r], r \in R^+ \), Define descriptive semigroups:
\[ T_{2^{-k}} = 2^{-k} (1 - 2^n A^\alpha_k)^{-1} + 1, \]
\[ T_0 = 1, \quad T_{1+s} = T_0 T_{1+s}, \quad T_{2^{-k}} = J 2^{-k}, s = J_2^{-k}. \]
Because \( (1 - 2^n A^\alpha_k)^{-1} \) and \( 2^{-k} A^\alpha_k = T_{2^{-k}} - 1 \) is the compression, \( \{ T_{2^{-k}}, t = J 2^{-k}, J = 0, 1, \ldots \} \) is the compression semigroup.

As \( A^\alpha_k = 2^{-k} (T_{2^{-k}} - 1) = A^\alpha_k (1 - 2^n A^\alpha_k)^{-1} \) is dissipative, we have \( \| A^\alpha_k T_{2^{-k}} f \| \leq \| A^\alpha_k f \| \), \( f \in \Omega \subset L^p(R^d, dx) \).

We show that \( \| T_{2^{-k}} u_n - T_{2^{-k}} u_m \| \leq \varepsilon \) for \( u_n^\alpha = q_n - 2^n q_m^\alpha, \quad q_n^\alpha = (1 - A^\alpha)^{-1} f_0, \quad q_m^\alpha = A^\alpha_k q_m^\alpha \), at \( n, m \geq n_0, \quad (\alpha, k) \in \varphi_0 \in \Psi_\infty, 0 \leq t \leq J 2^{-k} \leq r \). Really \( \| T_{2^{-k}} u_n^\alpha - T_{2^{-k}} u_m^\alpha \| = \|

= \sum_{i=1}^{\infty} \left( \left\| T_{2^{-k}} u_n^\alpha - T_{2^{-k}} u_m^\alpha \right\|^p - \left\| T_{2^{-k}} u_n^\alpha - T_{2^{-k}} u_m^\alpha \right\|^p \right) = \left( \left(1 - 2^n A^\alpha_k\right)^{-1} - T_{2^{-k}} u_n^\alpha \left(1 - 2^n A^\alpha_k\right)^{-1} T_{2^{-k}} u_m^\alpha \right) \right) \leq 0 \]

and

(\| T_{2^{-k}} u_n^\alpha - T_{2^{-k}} u_m^\alpha \| \leq 2^n \| A^\alpha_k T_{2^{-k}} u_n^\alpha \| \) then from \( \| A^\alpha_k T_{2^{-k}} f \| \leq \| A^\alpha_k f \| \), \( f \in \Omega \subset L^p(R^d, dx) \), follows that \( \| T_{2^{-k}} u_n^\alpha - T_{2^{-k}} u_m^\alpha \| \leq 4r \| q_n^\alpha \| \left( 2^n + 2^m + 2^{-k} \right) \) with \( j 2^{-k} \leq 2 \).

Denote \( t_k = j 2^{-k}, \quad j_k = [t 2^k], 0 \leq t \leq r \), where \( n \) – arbitrary fixed number with \( \mathcal{N} \), and \( \rho_j = \| T_{2^{-k}} u_n^\alpha - u_n (j 2^{-k}) \| \).

\[ T_{2^{-k}} u_n^\alpha = T_{2^{-k}} u_n^\alpha + \int_0^{1/2^k} A^\alpha_k T_{2^{-k}} u_n^\alpha ds = \left( j + 1 \right) 2^{-k} + \int_0^{1/2^k} A^\alpha_k u_n (j 2^{-k}) + \int_0^{1/2^k} A^\alpha_k (j 2^{-k} + s) ds \]

Because \( \lim_{\varphi_\infty} \rho_j \leq u_n (j 2^{-k}) - T_{2^{-k}} u_n^\alpha + \int_0^{1/2^k} (A^\alpha_k - A^\alpha_n) u_n (j 2^{-k} + s) ds + \int_0^{1/2^k} A^\alpha_n (j 2^{-k} + s) - A^\alpha_n T_{2^{-k}} u_n^\alpha ds \leq \)

\[ \rho_{j+1} \leq u_n (j 2^{-k}) - T_{2^{-k}} u_n^\alpha + \int_0^{1/2^k} (A^\alpha_k - A^\alpha_n) u_n (j 2^{-k} + s) ds + \int_0^{1/2^k} A^\alpha_n (j 2^{-k} + s) - A^\alpha_n T_{2^{-k}} u_n^\alpha ds \leq \]
\[ \rho_j + 2t^k \varepsilon + 2t^k(\rho_j + 2t^k \| q_0^k \|) \leq \rho_j (1 + 2t^{k-2}) + 2t^{-k} \varepsilon \quad \exists \varphi \in \Psi^* \]

is performed for inequality
\[ 2t^k \| q_0^k \| \leq 2t^k \varepsilon . \]

Then, using induction, we get:
\[ \rho_j \leq \rho_0 e^{-2t^{k+j}} + 2t^{k+j} \varepsilon \leq (\| q_0^k - q_0 \| 2t^{k-2} \| q_0^k - q_0^k \|)^{e^{-2t^{k+j}}} + 2t^{k+j} \varepsilon \]

when \( 0 \leq j \leq r \), therefore
\[ \lim T_{\tau,\alpha} u_n^\alpha = u_n(t) . \]

As
\[ \lim_{n \to \infty} A_n^\alpha (1 - 2t^{k-n} A_n^\alpha)^{-1} q = A_n^\alpha q \quad \text{when} \quad q \in D(A_n^\alpha) = \Omega \subset L^p(R', d'x) , \]

for fixed \( \alpha \), we have
\[ \lim_{n \to \infty} T_{\tau,\alpha} u_n^\alpha - T_{\tau,\alpha} u_n^\alpha = 0 \quad \text{and} \quad \{ u_{n,\alpha}^\alpha \}_{n=\alpha} \]

is a relatively compact set in \( \Omega \subset L^p(R', d'x) \).

When \( s = 2t^2, t = j2^{-k}, s + t = (i + j)2^{2k} \in [0, r] \) we get
\[ \| u(s + t) - T_{i} f \| \leq \| u(s + t) - u_{n}(s + t) \| + \| u_{n}(s + t) - T_{\tau,\alpha} u_n^\alpha \| + \| T_{\tau,\alpha} u_n^\alpha - T_{\tau,\alpha} u_n^\alpha \| + \| T_{\tau,\alpha} u_n^\alpha - T_{\tau,\alpha} u_n^\alpha \| + \| f \| + 6 \varepsilon \leq \| u(s) - f \| + 6 \varepsilon , \]

since
\[ \| u(s) - T_{\tau,\alpha} u_n^\alpha \| \leq \| u(s) - u_{n}(s) \| + \| u_{n}(s) - T_{\tau,\alpha} u_n^\alpha \| + \| T_{\tau,\alpha} u_n^\alpha - T_{\tau,\alpha} u_n^\alpha \| \leq 3 \varepsilon . \]

Because of the uniform continuity \( T_{i} f \) and \( u(t) \) indeed there is an inequality
\[ \| u(s + t) - T_{i} f \| \leq \| u(s) - f \| \quad \text{with} \quad f \in D(T_i) , \quad s, t, s + t \in [0, r] . \]

In other words \( \{ \mathcal{T}_i \} \) is a contraction semigroup, but this leads to a contradiction with maximality of semigroup \( \{ T_i \} \), so we get a contradiction. We show that the region \( D(T_i) \) is not contained in some closed hyperplane in \( L^p(R', d'x) \).

Suppose \( D(T_i) \subset \{ f \in L^p(R', d'x) : (f, e) = M \} \) for some \( e \in L^p(R', d'x) \) and \( \| e \| = 1 \).

Let \( S_i(f + e) = T_i f + e \) at \( f \in D(T_i) \), then \( \{ S_i \} \) is also a compression semigroup and the definitional domain of \( S_i \) is the set \( e \cup D(T_i) \), which has an empty intersection with \( D(T_i) \), so
\[ \mathcal{T}_i = \{ T_i f \} , \quad \text{npu} \quad f \in D(T_i) \]

is an extension of \( \{ T_i \} \), but \( \{ \mathcal{T}_i \} \) is a compression semigroup, i.e., there is a conflict with maximality \( \{ T_i \} \). Theorem 1 is proved.

**Theorem 2.** The closure of the set \( D(A) \), where \( A \) is the maximum dissipative operator is a set which is convex in \( L^p(R', d'x) \).

**Proof.** We will use contradiction method. Let \( q, \Psi \in D(A) \) and \( f = \mu q - (1 - \mu)\Psi \) when \( 0 < \mu < 1 \). Suppose that \( f \notin D(A) \), then put \( q_{\lambda} = q - \lambda q_{\lambda} \), with \( q_{\lambda} \in Aq \). Using statement 8, we get
\[ \| (1 - \lambda A)^{-1} f - q \| = \| (1 - \lambda A)^{-1} f - (1 - \lambda A)^{-1} q \| \leq \| f - q \| , \]

when \( 0 < \lambda < 1 \), we have that
\[ \leq \| f - q \| + \| f - \Psi \| = \| q - \Psi \| , \]

and thus we get that
\[ \lim_{\lambda \to 0} (1 - \lambda A)^{-1} f = f , \]

but this contradicts the...
Theorem 3. 1) The maximum compression semigroup \( \{T_t\} \) has tightly defined generator and it has been generated by a maximum dissipative operator. 2) If the maximum dissipative operator is single valued, then the semigroup generated by this operator is a maximal contraction semigroup.

Proof. Proposition 1) is a consequence of previous theories and assertions. Indeed local generator \( \{T_t\} \) \( A_0 \) is consistently expressed in \( D(T_t) \), maximum dissipative extension \( A \) operator \( A_0 \) generates a compression semi-group \( \{S_t\} \), then semigroup \( \{S_t\} \) is an extension of semigroup \( \{T_t\} \), but out of maximality \( \{T_t\} \) we get \( \{S_t\} = \{T_t\} \). 1) has been proven.

Proposition 2) prove by contradiction. Let the operator \( A \) generates semigroups \( \{T_t\} \) and \( \{S_t\} \) – the maximum extension \( \{T_t\} \). Assume the opposite \( D(S_t) \supsetneq D(T_t) \) and \( D(S_t) \neq D(T_t) \), use 1) generator \( \{S_t\} \) let \( \tilde{A} \) be consistently defined in \( D(S_t) \).

Because of closure \( D(T_t) \) there is an element \( f \in D(\tilde{A}) \) and \( f \not\in D(T_t) \subset L^p(R^d, d^1 x) \) using the maximum dissipativity \( A \) get that exist \( q_\lambda = (1 - \lambda A)^{-1} f \), \( \forall \lambda > 0 \) and \( \lim_{\lambda \to 0} q_\lambda = q \in [D(A)] \subset L^p(R^d, d^1 x) \). Since \( q_\lambda \in D(A) \), \( T_t q_\lambda \) weakly differentiable on \( t \), and \( \lambda - \lim_{h \to 0} A_h q_\lambda = A q_\lambda \) and \( A q_\lambda = q_\lambda - f \frac{f}{\lambda} \) thus \( \lim_{h \to 0} \frac{\lambda}{h} \langle T_h q_\lambda - q_\lambda, (q_\lambda - f) |q_\lambda - f|^{\nu - 2} \rangle \geq \|q_\lambda - f\|^\nu \),

\[ \langle S_h f - f, (q_\lambda - f) |q_\lambda - f|^{\nu - 2} \rangle = \langle S_h f - T_h q_\lambda, (q_\lambda - f) |q_\lambda - f|^{\nu - 2} \rangle + \|T_h q_\lambda - q_\lambda, (q_\lambda - f) |q_\lambda - f|^{\nu - 2} \rangle \]

since \( \|T_h q_\lambda - S_h f \| \leq \|S_h q_\lambda - S_h f \| \leq \|q_\lambda - f\| \).

That is, we have \( \lim_{h \to 0} \frac{\lambda}{h} \langle S_h f - f, (q_\lambda - f) |q_\lambda - f|^{\nu - 2} \rangle \geq \|q_\lambda - f\|^\nu \), directing \( \lambda \) to zero, we obtain a contradiction

\[ \lim_{h \to 0} \frac{\lambda}{h} \langle S_h f - f, (q_\lambda - f) |q_\lambda - f|^{\nu - 2} \rangle = \langle \tilde{A} f, (q_\lambda - f) |q_\lambda - f|^{\nu - 2} \rangle \rightarrow \langle \tilde{A} f, (q_\lambda - f) |q_\lambda - f|^{\nu - 2} \rangle \] \( \|q_\lambda - f\|^\nu \)

\[ \frac{\lambda}{h} \rightarrow \infty \]. Theorem is proved.

Sequences of nonlinear semigroups in spaces \( L^p(R^d, d^1 x) \). established that: Cauchy problem \( \frac{d}{dt} u(t) \in A u(t), \ u(t) \in L^p(R^d, d^1 x), \ t \in [0, t_0] \), \( u(0) = u_0 \), with each \( u_0 \in D(A) \) has only a weak solution if the operator \( A \colon L^p(R^d, d^1 x) \rightarrow L^p(R^d, d^1 x) \) is maximal dissipative operator; Let \( A_0 \) be consistently defined generator of compression semigroup \( T_t \), while its maximum dissipative expansion \( A \) generates the same compression semi-group \( T_t \); In addition, it was found that the generator of nonlinear compression semigroup is consistently defined in \( L^p(R^d, d^1 x) \).

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Theorem 4. Let \( \{ T^n_t : t \geq 0, n \in \mathbb{N} \} \) be a sequence of nonlinear semigroups, satisfying condition: \( \| T^n_t f - T^n_t g \| \leq e^{\alpha t} \| f - g \| \) and the sequence of operators \( \{ A_n : n \in \mathbb{N} \} \) is a generator of nonlinear semigroups \( \{ T^n_t : t \geq 0, n \in \mathbb{N} \} \) sequence and there is a sequence of numbers \( \mu^n \in \left( 0, \frac{1}{\omega} \right) \), so that \( R \left( 1 - \mu^n A_n \right) = L^p \left( \mathbb{R}^d, \mathbb{R}^d \right) \). Denote the border of elements from \( L^p \left( \mathbb{R}^d, \mathbb{R}^d \right) \) \( \lim_{n \to \infty} A_n f \) with \( A f \) as a border in \( L^p \left( \mathbb{R}^d, \mathbb{R}^d \right) \)-norm.

Then the closure of \( A \) in \( L^p \left( \mathbb{R}^d, \mathbb{R}^d \right) \) norm, which we denote with \( [A] \) generates nonlinear semigroup \( \{ T^\alpha_t : t \geq 0 \} \), which can be defined as \( L^p \left( \mathbb{R}^d, \mathbb{R}^d \right) \)-uniform border:

\[
T_t f = \lim_{n \to \infty} T^n_t f \quad \text{on any finite interval } t \in [0, t_0].
\]

In addition, the semigroup \( \{ T^\alpha_t : t \geq 0 \} \) is the only one in the class of semigroups which satisfies the following conditions:

1) for any element \( f \in D(A) \) function \( T_t f \) is strongly absolutely continuous on any finite interval;

2) for any element \( f \in D(A) \) for all \( t \geq 0 \) \( T_t f \in D(A) \) and \( D^t T_t f = A_n T_t f \) and \( A_n T_t f \) is continuous by the norm \( L^p \left( \mathbb{R}^d, \mathbb{R}^d \right) \) at \( t \geq 0 \) by \( t \).

3) for any element \( f \in D(A) \), there is a strong continuous derivative \( \frac{d}{dt} T_t f = A_n T_t f \) except perhaps countable number of points.

**Proof.** The proof methods used are similar to those that were used above. For convenience and to avoid confusion code sequence is set in brackets, i.e. semigroup \( T^n_t \) generator \( A_n \) will continue to be marked as \( T^{(n)}_t \) and \( A^{(n)} \) respectively.

Let the elements \( f, g \in D(A) \subseteq L^p \left( \mathbb{R}^d, \mathbb{R}^d \right) \)

\[
\left\{ A^{(n)} f - A^{(n)} g, (f - g) \right\}^{p-2} = \lim_{\alpha \to 0} \left\{ \frac{T^{(n)}_\alpha f - f}{\alpha} - \frac{T^{(n)}_\alpha g - g}{\alpha}, (f - g) \right\}^{p-2} \leq \omega \| f - g \|,
\]

therefore the operator \( A^{(n)} - \omega I \) is a dissipative operator. Since \( R \left( 1 - \mu_n A^{(n)} \right) = L^p \left( \mathbb{R}^d, \mathbb{R}^d \right) \), then for \( \eta_n = \mu_n \frac{1}{1 - \mu_n \omega} \) and every \( n \) the equality \( R \left( 1 - \eta_n \left( A^{(n)} - \omega I \right) \right) = L^p \left( \mathbb{R}^d, \mathbb{R}^d \right) \) is true, so \( A^{(n)} - \omega I \) is the maximum dissipative operator. Then fix sequence index, one that is in parentheses, and use previous results, which is always possible when \( m > \omega \) then there is \( \left( 1 - \frac{A^{(n)}}{m} \right)^{-1} \) for this operator the evaluation is true for \( f, g \in L^p \left( \mathbb{R}^d, \mathbb{R}^d \right) \)

\[
\left\| \left( 1 - \frac{A^{(n)}}{m} \right)^{-1} f - \left( 1 - \frac{A^{(n)}}{m} \right)^{-1} g \right\| \leq \left( 1 - \frac{\omega}{m} \right)^{-1} \| f - g \|.
\]
Let us put by definition $A_m^{(n)} = m \left[ 1 - \frac{A^{(n)}}{m} \right]^{-1}$. So nonlinear operators $A_m^{(n)}$ are generators of nonlinear semigroups $\{ T_{mt}^{(n)} : t \geq 0, n \in \mathbb{N} \}$ and for these semigroups for $f, g \in L^p \left( \mathbb{R}^l, d^l x \right)$ assessment $\| T_{mt}^{(n)} f - T_{mt}^{(n)} g \| \leq e^{\alpha t} \| f - g \|$ is true. For elements $f \in [D \left( A^{(n)} \right)]$ a border $T_t^{(n)} f = \lim_{m \to \infty} T_{mt}^{(n)} f$, $t \geq 0$ exists.

Define the Cauchy difference for semigroups $\{ T_{mt}^{(n)} : t \geq 0, n \in \mathbb{N} \}$ sequences using index $m : T_{mt}^{(n)} f - T_{kt}^{(n)} f$, $t \geq 0$, and mark it $P_{mk}^{(n)}$, that is $P_{mk}^{(n)} f = T_{mt}^{(n)} f - T_{kt}^{(n)} f$, $t \geq 0$, and because the way $P_{mk}^{(n)}$ acts is important only for large $k, m$, it can be assumed that $k, m \geq p \omega$, thus the "tail" of $P_{mk}^{(n)}$ sequence is investigated.

Using estimates obtained and given the already introduced symbols, we get for $\alpha \geq 0$: $\| P_{mk}^{(n)} f \| \leq 2 p e^{2 \omega m} \| A^{(n)} f \| \alpha$, 

$$
\left\| P_{mk}^{(n)} f - \left( 1 - \frac{A^{(n)}}{m} \right) T_{mt}^{(n)} f + \left( 1 - \frac{A^{(n)}}{k} \right) T_{kt}^{(n)} f \right\| \leq \left\| A_m^{(n)} T_{mt}^{(n)} f \right\| + \left\| A_k^{(n)} T_{kt}^{(n)} f \right\| \leq \left( \frac{1}{m - \omega} + \frac{1}{k - \omega} \right) e^{2 \omega m} \| A^{(n)} f \|,$$

then if we denote

$$\mathcal{K}_{mk}^{(n)} (\alpha) = \text{const}(p) e^{p \omega} \inf \{ \| g \| : g \in A^{(n)} \} \times$$

\begin{align*}
\int_0^\beta \left\| T_{mt}^{(n)} f - T_{kt}^{(n)} f \right\|^p \| T_{mt}^{(n)} f - T_{kt}^{(n)} f \|^{p - 2} - \\
\left\| \left( 1 - \frac{A^{(n)}}{m} \right) T_{mt}^{(n)} f - \left( 1 - \frac{A^{(n)}}{k} \right) T_{kt}^{(n)} f \right\|^{p - 2} \right) d \eta + \\
+ \left(c_1(p)\right)^p \omega \left( \frac{1}{m - \omega} + \frac{1}{k - \omega} \right)^p e^{2 \omega p} \beta \| A^{(n)} f \|^p,
\end{align*}

we get similar assessment $L^p \left( \mathbb{R}^l, d^l x \right)$ of the norm $P_{mk}^{(n)}$ in the segment $\alpha \in [0, \beta]$: $\| P_{mk}^{(n)} \| \leq \mathcal{K}_{mk}^{(n)} (\beta) e^{p \omega}$.

Then we prove that the convergence in the limit $T_t^{(n)} f = \lim_{m \to \infty} T_{mt}^{(n)} f$, $t \geq 0$, of any number $\alpha \in [0, \beta]$ is uniform regarding $n$.

**Proof.** Fix an arbitrary number $\alpha \in [0, \beta]$ and element $f \in D \left( A^{(n)} \right)$. Since $\lim_{n \to \infty} A^{(n)} f = Af$, there is a natural number $n_0$ and the number $M > 0$ that at $n > n_0$,
$f \in D \left( A^{(n)} \right)$, and $\| A^{(n)} f \| \leq M$, that for large indexes of norm sequence generators are uniformly bounded on each element.

Obviously, the set:

$$B = \{ \left( T^{(n)}_{mn} f - T^{(n)}_{kq} f \right) \left| T^{(n)}_{mn} f - T^{(n)}_{kq} f \right|^{p-2} \left( 1 - \frac{A^{(n)}_m}{m} \right)^{-1} T^{(n)}_{mn} f - \left( 1 - \frac{A^{(n)}_k}{k} \right)^{-1} T^{(n)}_{kq} f : \eta \in \{0, \beta\}, n > n_\eta, m > p\omega, k > p\omega \}
$$

is limited.

Using the previous inequalities we find out that for every number $\varepsilon > 0$ there is such a number $\delta(\varepsilon) > 0$, that for elements $f, g \in B$ and $\| f - g \| < \delta(\varepsilon)$ inequality

$$\left\| f \right\|^{p-2} - \varepsilon \left\| g \right\|^{p-2} \leq \frac{\varepsilon^p}{2p^2 M \beta e^{\mu \omega}} \text{const}
$$

is true.

We choose among numbers $k, m > k_\eta > \omega p$ large enough, so they fit the inequality

$$2 \frac{p^2 M e^{\mu \omega}}{(k_\eta - \omega) c_1(p)} \leq \min \left( \delta, \frac{\varepsilon}{\sqrt{p^2 \omega \alpha}} \right),$$

and get at $\alpha \in \{0, \beta\}$ and

$$\left\| P^{(n)}_{mk} f \left( 1 - \frac{A^{(n)}_m}{m} \right)^{-1} T^{(n)}_{mn} f + \left( 1 - \frac{A^{(n)}_k}{k} \right)^{-1} T^{(n)}_{kq} f \right\| \leq 2 \frac{M e^{\mu \omega}}{(k_\eta - \omega) \lambda} \leq \delta$$

thus at $n > n_\eta$ we have:

1. $\left\| P^{(n)}_{mk} f \left( 1 - \frac{A^{(n)}_m}{m} \right)^{-1} T^{(n)}_{mn} f + \left( 1 - \frac{A^{(n)}_k}{k} \right)^{-1} T^{(n)}_{kq} f \right\|^{p-2} \leq \frac{\varepsilon^p}{2p^2 M \beta e^{\mu \omega}} \text{const}$

Let us show that for $n > n_\eta$ inequality $K^{(n)}_{mk} (\beta) \leq \varepsilon^p$. Indeed we have estimates

$$\text{const}(p) e^{\mu \omega} \inf \{ \| g \| : g \in A^{(n)} \} \times \frac{\varepsilon^p}{2} \left\| \left( T^{(n)}_{mn} f - T^{(n)}_{kq} f \right) \left| T^{(n)}_{mn} f - T^{(n)}_{kq} f \right|^{p-2} \right\|$$

$$\leq \frac{\varepsilon^p p^2 M \beta e^{\mu \omega}}{2p^2 M \beta e^{\mu \omega}} ;$$

$$c_1(p) \omega \left( \frac{1}{m - \omega} + \frac{1}{k - \omega} \right)^p e^{2p\mu \omega} \| A^{(n)} f \|^p \leq (c_1(p))^p \left( \frac{p^2 M e^{\mu \omega}}{k_\eta - \omega k_\eta} \right)^p \beta \leq \frac{\varepsilon^p}{2} .$$

So, for $k, m > k_\eta$ a valid assessment is $\sup_{\eta \in \{0, \beta\}, n > n_\eta} \left\| T^{(n)}_{mn} f - T^{(n)}_{kq} f \right\| \leq e^{\mu \omega} \varepsilon$. Since

$$\left[ R \left( 1 - \mu_\omega A \right) \right] = L^p \left( R^l, d^l \right)$$

and operator $A - I \omega$ is dissipative as the border of operators
\( A^{(n)} - 1 \omega \) at \( n \to \infty \), we get the assertion of uniform convergence of semigroups 
\[
T_{i}^{(n)} f = \lim_{m \to \infty} T_{m}^{(n)} f, \quad t \geq 0
\]. To complete the proof we use Lemma 4 [6, 7].

**Lemma 4.** Let \( m > \omega \) then the operator \( \left( 1 - \frac{A}{m} \right)^{-1} \) has a unique extension \( B_{m} \) which is determined in the entire area \( L^{p}(R^{i}, d^{i}x) \) and for which in the entire \( L^{p}(R^{i}, d^{i}x) \) assessment
\[
\|B_{m} f - B_{m} g\| \leq \frac{\|f - g\|}{1 - m^{-1} \omega}
\] is correct. In addition \( B_{m} = \left( 1 - \frac{[A]}{m} \right)^{-1} \) and the operator \( (\frac{[A]}{1 - \omega})^{-1} \) is a maximal dissipative operator.

**Theorem 4.** (on the generalized Cauchy problem in \( L^{p}(R^{i}, d^{i}x) \)). Generalized Cauchy problem
\[
\frac{d}{dt} u(t) \in Au(t), \quad u(t) \in L^{p}(R^{i}, d^{i}x), \quad t \in [0, t_{0}], \quad u(0) = u_{0},
\] where \( A : L^{p}(R^{i}, d^{i}x) \to L^{p}(R^{i}, d^{i}x) \), each has at every \( u_{0} \in D(A) \) only a single weak solution.

**Conclusions.** We have constructed operator functions of exponential type, investigated the link between these operator functions and generalized initial Cauchy problem for equations of parabolic type. The existence of a solution of the generalized Cauchy problem for equations of parabolic type has been proven. The results can be generalized to classes of differential operators of more general type operating in certain functional spaces.

**References**

Sequences of semigroups of nonlinear operators and their applications to study the cauchy problem for parabolic equations

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ПОСЛІДОВНОСТІ НАПІВГРУП НЕЛІНІЙНИХ ОПЕРАТОРІВ ТА ЇХ ЗАСТОСУВАННЯ ДЛЯ ДОСЛІДЖЕННЯ ЗАДАЧІ КОШІ ДЛЯ РІВНЯННЯ ПАРАБОЛІЧНОГО ТИПУ

Микола Яременко

Національний технічний університет України «Київський політехнічний інститут імені Ігоря Сікорського», Київ, Україна

Резюме. Розглянуто операторні функції експоненціального типу, досліджено зв’язок між такими функціями (напівгрупами) та задачами Коші для диференціального параболічного рівняння. Встановлено умови, за яких напівгрупа буде асоційовано з задачею Коші, досліджено послідовності напівгруп та їх збіжність до певної напівгрупи. Розглянуто максимальні дисипативні оператори та максимальні напівгрупи, а також задачу про існування розв’язку нелінійних диференціальних рівнянь у частинних похідних параболічного типу з вимірними коефіцієнтами, нелінійний доданок яких задовольняє умови форм – обмеженості коефіцієнтів.

Ключові слова: квазілінійні диференціальні рівняння, дисипативні оператори, метод форм, напівгрупа, максимальні оператори, послідовності напівгруп.

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