Estimating the Periodicity in the Structure of Stochastic Fields

Ya. P. Dragan, N. R. Krivaya, and B. I. Yavorskii

Ternopol Institute of Instrument Making, ul. Rus'ka 56, Ternopol, 282001, Ukraine

In many (we can even argue that in the majority of) applications of radio engineering and radiophysics (exploration of the space, sea, interior of the earth; seismic geophysical investigations; etc.), input data vary in space and time. Processing of such data, first, at the stage of data sampling and transmission and then in the process of analysis, should take these variations into account. Optimal processing conditions in space and time are usually considered separately from each other. Optimizing spatial processing, one takes into consideration the properties of amplitude distribution of a certain physical quantity in space and time for the chosen method of detection and the given design of detectors. Temporal optimization implies analysis of the detected signal. However, such an approach cannot guarantee optimal properties of the system as a whole because optimizing the entire system requires the consideration of a unified criterion. Such a criterion should be chosen within the framework of a model of the problem being solved by the system (detection, estimation of parameters, classification, or recognition). Taking into account the specific features of (digital) signal processing, physics of an object, treatment of the input signal as a multidimensional random process, and interpretation of the system for spatial and temporal processing as a multichannel system, we should choose and develop appropriate models.

In such a situation, it is natural to choose the model of the signal (the spatial–temporal field) in the form of a function of many variables whose arguments are defined as the coordinates of the point specified by a vector $\mathbf{r} = (x, y, z)$ and the time t. In correlation theory, along with the mathematical expectation $m(t, \mathbf{r}) = E\xi(t, \mathbf{r})$, where E denotes averaging over the distribution, a field is characterized by the spatial–temporal correlation function (STCF),

$$r(t_1, t_2; \mathbf{r}_1, \mathbf{r}_2) = E\xi(t_1, \mathbf{r}_1)\overline{\overset{0}{\xi}(t_2, \mathbf{r}_2)},$$

where ^{0} indicates a centered random quantity. The field is referred to as stationary if this function depends only on the difference $t_1 - t_2 = u$, and the field is called uniform if this function depends on $\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{v}$ (in these situations, the mathematical expectation is independent of parameters t and $\mathbf{r}$, respectively). The field is isotropic if the STCF depends only on $|\mathbf{v}|$. The spectral expansion is defined as the representation of the field as a superposition of plane monochromatic waves of the form

$$p(t, \mathbf{r}) = \exp\{i(t - \mathbf{k}\mathbf{r})\},$$

where $\mathbf{k}$ is the wave vector in the direction orthogonal to the wave front. In other words, the spectral expansion can be written as

$$\xi(t, \mathbf{r}) = \int\limits_{R^x} \exp[i(t - \mathbf{k}\mathbf{r})]Z(d\lambda, d\mathbf{v}), \qquad (1)$$

where $Z(\Delta_\lambda, \Delta_r)$ is the stochastic measure. This expansion of the field corresponds to the representation of the STCF in the form

$$r(t_1, t_2; \mathbf{r}_1, \mathbf{r}_2) = \int\limits_{R^x} \exp[i(\lambda_1 t_1 - \lambda_2 t_2 - \mathbf{k}_1\mathbf{r}_1 + \mathbf{k}_2\mathbf{r}_2)]F(d\lambda_1, d\lambda_2; d\mathbf{r}_1, d\mathbf{r}_2).$$

For a stationary and uniform field, we have

$$r(t_1, t_2; \mathbf{r}_1, \mathbf{r}_2) = R(t_1 - t_2; \mathbf{r}_1 - \mathbf{r}_2) = R(u, \mathbf{v}). \qquad (2)$$

In this case, which develops the concept of a conventional stationary stochastic process of a single variable, we can apply, after obvious modifications, methods of processing developed for stationary stochastic processes of a single variable.

The next step toward selecting practically important, physically significant, and mathematically constructive stochastic fields is associated with introducing periodically distributed and periodically correlated random fields. Such fields extend the rhythm of natural phenomena and models of this rhythm to the corresponding types of periodical nonstationary processes. In turn, these processes form a subclass of stationarized (analogs of stationary processes in the restricted sense) and energy classes (analogs of second-order processes [2, 3]). The theory of such processes provides a background for developing methods of studying the periodicity in the structure of random fields. Specifically, we will define periodically correlated fields as fields whose STCFs feature the invariance with respect to joint shifts by quantities T and $\mathbf{L} = (L_x L_y, L_z)$ represented by fixed numbers, which are referred to as correlation periods. In other words, we have

$$r(t_1 + T, t_2 + T; \mathbf{r}_1 + \mathbf{L}, \mathbf{r}_2 + \mathbf{L}) = r(t_1, t_2; \mathbf{r}_1, \mathbf{r}_2). \qquad (3)$$

Clearly, if the property of invariance holds for any shifts, the above relation is reduced to formula (2), and

Received October 1, 1995

we deal with a stationary isotropic field. Using formula (3), we derive

$$F(d\lambda_1, d\lambda_2; d\mathbf{r}_1, d\mathbf{r}_2) = \sum_{k, p = -\infty, \infty} \delta(\lambda_1 - \lambda_2 + kT) \times \delta(\mathbf{r}_1 - \mathbf{r}_2 + p\mathbf{L}) F_{kp}(d\lambda_1, d\mathbf{r}_1).$$

This property demonstrates that the considered field can be decomposed into stationary isotropic components coupled by stationary isotropic relations.

We can describe random fields of the above-specified class by applying, with appropriate modification, all methods of analysis developed for periodically nonstationary random processes. Specifically, we can use the inphase method, which is based on a sequence of counts separated by the correlation period, and the method of filters, which uses spectral fragments with frequency bands $\Delta = \frac{2\pi}{T}$ and $\mathrm{R} = \frac{2\pi}{\mathrm{L}}$. These methods reduce the field to a combination of stationary isotropic components. Furthermore, we can apply the component method, which implies estimating components of Fourier characteristics. These methods are applicable if the correlation periods are known.

A substantially different situation occurs in the opposite case that is closely related to the general problem of revealing a hidden periodicity. This situation was originally considered by A. Schuster. In this case, we deal with the problem of determining the correlation period.

A natural method for determining the correlation period is to generalize the well-known Bui-Ballot scheme for estimating the period of a periodic function. According to this approach, we should estimate characteristics of a periodically correlated random field by applying one of the methods for each trial value of the correlation period $T_p = T_0 + ph, p = \overline{0, M-1}$, under the assumption that the segment $[T_0, (M-1)h + T_0]$ includes this period. Then, the criterion of the best estimate of the period is defined as the minimum smoothing of the estimate of the probabilistic characteristic of the field $\chi_M(T_p)$ calculated for an N-dimensional sample, i.e., as the maximum depth of the relief of this estimate. To introduce a measure of the relief depth, we can use the amplitude or any variation of this estimate.

Note that the assumption that the correlation period is a random quantity, which is widely encountered in literature, is incorrect. Indeed, using this assumption, we, in fact, accept another model corresponding to a mixture of periodically correlated random processes [4] or to twice-stochastic processes [1]. Determining probabilistic characteristics in such a situation would require averaging over the distribution of the period. However, in the latter case, the resulting process cannot be considered as periodically correlated, which is illustrated by a counterexample of a process $\xi(t) = \cos\alpha t$, where α is a random quantity that is uniformly distributed within the segment $[-1, 1]$. Obviously, the mathematical expectation $m_\xi(t) = \frac{\sin t}{t}$ and the covariances $r(t, s) = \frac{\sin(t+s)}{t+s} + \frac{\sin(t-s)}{t-s} - \frac{\sin t \sin s}{ts}$ for such a process are aperiodic. Therefore, the meaning of the results obtained under this assumptions is not clear.

REFERENCES

1. Goodman, J., *Statisticheskaya Optika* (Statistical Optics), Moscow: Mir, 1988.
2. Dragan, Ya.P., Vasil'ev K.K., *et al.*, *Prikladna teoriya vipadkovikh protsesiv i poliv* (Applied Theory of Random Processes and Fields), Ternopol': TPI, 1993.
3. Dragan, Ya.P., Stationary Random Processes and the Energy Concept in the Theory of Signals, *Tez. dokl. regional'noi nauchno-tekhnicheskoi konferentsii "Izmerenie kharakteristik sluchainykh signalov s primeneniem mikromashinnykh sredstv"* (Abstracts of Papers of the Regional Scientific and Technical Conf. "Measurement of Characteristics of Random Signals Using Microcomputer Means"), Novosibirsk: NETI, 1988, Pt. 1, pp. 58–59.
4. Patrik, E., *Osnovy teorii raspoznavaniya obrazov* (Fundamentals of the Theory of Pattern Recognition), 1980, Moscow: Sov. Radio.