MULTIFRACTAL APPROACH TO THE DESCRIPTION AND WAVELET ANALYSIS OF THE FATIGUE DAMAGE

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Abstract A phenomenological model for the multifractal description of the fatigue damage accumulation has been introduced. The multifractal approach gives probabilistic evidence for the existence of a constructive process hidden in the temporal pattern of damage accumulation. It is shown that a series of Bernoulli trials results in multiplicative bi- or polynomial process that recursively generates the multifractal probability measure. Connection between parameter of the Bernoulli trials and multifractal spectrum is considered. A technique for revealing the multifractal properties of the damage accumulation is proposed. For the approbation of the technique, a computer simulation study has been done. The wavelet transform has been used for revealing the intrinsic temporal structure of data sets obtained from numerical simulation, tests, empirical observations and measurements.

1. Introduction

The purpose of the paper is to introduce an application of the multifractal theory in the reliability engineering, risk analysis and others fields where we deal with series of events of damage accumulation. We do not know whether some temporal pattern is hidden in an apparently disordered set of events. The multifractal theory is a good basis for describing the event sequence in time. It can provide a deeper understanding the nature of the event flow.

In reliability engineering for instance, the Weibull distribution is one of the most widely used distribution because through the appropriate choice of parameters a variety of failure rate behaviors can be modeled. The two-parameter Weibull distribution assumes that the failure rate $\lambda(t)$ is in the form of a power law [1]:

$$\lambda(t) = \alpha\, \lambda^{\alpha}\, t^{\alpha-1} \tag{1}$$

for all $t \geq 0$, where α and λ are positive and are referred to as the shape and scale parameters of the distribution respectively. Such power laws, with integer or fractional exponents, are in fact endless source of self-similarity or precisely, self-affinity. They can be qualified as self-affine functions since their graphs are similar to themselves when transformed by anisotropic dilations. Deeper insight into damage accumulation, its prediction and prevention is to be gained by using the multifractal approach.

In this paper, we present a technique for revealing the multifractal properties of the damage accumulation process. For the approbation of the technique, a computer simulation study has been done. Wavelet analysis was carried out in order to verify the fractality of data sets obtained from numerical simulations, tests and operation. The continuous wavelet transform of empirical data on damage accumulation provides probabilistic evidence that a multifractal description is appropriate.

2. Mathematical model. We refer to the failure sequence associated with damage accumulation as a succession of appearance of the failure events in time. A mathematical construction that represents an event sequence as a set of the random points on the time scale is referred to as a stochastic point process. Among discrete processes, point processes are widely used in engineering to describe the random events occurring in a system during its lifetime (e.g., failures, jumps of damage accumulation, terminations of repair, demand arrivals, etc.). The point process can either be modeled as a list of impulses (jumps) located at times where events occur or as a count process, similar in a sense to the “devil staircase” fractal. Such a process may be called fractal when a number of the relevant statistics of the point process exhibits scaling with related scaling exponents, indicating that the represented phenomenon contains clusters of events over a relatively large set of time scales. This scaling leads naturally to power-law behavior.

Let S be a sample of events of limited size N_0 during the specified period of time $[0, \tau_{max}]$. The process time history for the sample S is represented by a sequence of idealized impulses (jumps) of vanishing width, located at specified moments of the event time $\tau(i)$, $(i = 1,..., N_0)$. Further, let us rescale the time $\tau(i)$ of every i-th member of sample S on the maximum value τ_{max} $t(i) = \tau(i) / \tau_{max}$. Now we can consider the event distribution on the unit interval of time $T = [0, 1]$. In order to characterize this distribution we divide the unit interval into temporal subintervals of duration $\Delta t = 2^{-n}$. So $N = 2^n$ subintervals are needed to cover interval T, where n is the number of generation in the binary subdivision of the temporal interval T. Let us label the subintervals by the index $j = 0, 1, 2, \dots, N-1$. The distribution of the sample population over the temporal interval is specified by the numbers, N_j, of members of the sample S in the j-th subinterval. We use the fraction of the total population $\mu_j = N_j / N_0$ as a probabilistic measure for the content in subinterval Δt_j. The set Ω of such probabilistic measures $\Omega = \{\mu_j\}_{j=0}^{N-1}$ presents a complete description of the event's distribution on interval T at stated resolution Δt.

Now let us consider a case that satisfies the Bernoulli trial conditions. In our interpretation an event of interest is the failure, associated with damage accumulation, occurred on the first half of interval T with probability p. The series of Bernoulli trials with parameter p is a sequence of generic independent trials in which there are only two outcomes, and probability p remains the same for all generations of the binary subdivision process of the interval T.

In the case of Bernoulli trials the probabilistic measure μ is recursively generating by a multiplicative binomial process (MBP) [2,3]. The process provides an example of a probability which has a rich asymptotic structure and is, in modern terms, multifractal [4]. In fact, the binomial multifractal measure is a product of the multiplicative cascade, which attributes probabilities, to the dyadic temporal subintervals of the interval T.

3. Wavelet transform of a measure. Wavelet transforms play an important role in the study of self-similar and self-affine measures. The continuous wavelet transform (CWT) $W_{ab}\{\mu(t)\}$ of a measure $\mu(t)$

$$W_{ab}\{\mu(t)\} = \int_{Supp\,\mu} \psi_{ab}(t)\, d\mu\,(t) \tag{2}$$

is defined in terms of projections of $\mu(t)$ onto a family of functions of the form

$$\psi_{ab}(t) = |a|^{-1/2}\, \psi\left(\frac{t-b}{a}\right) \tag{3}$$

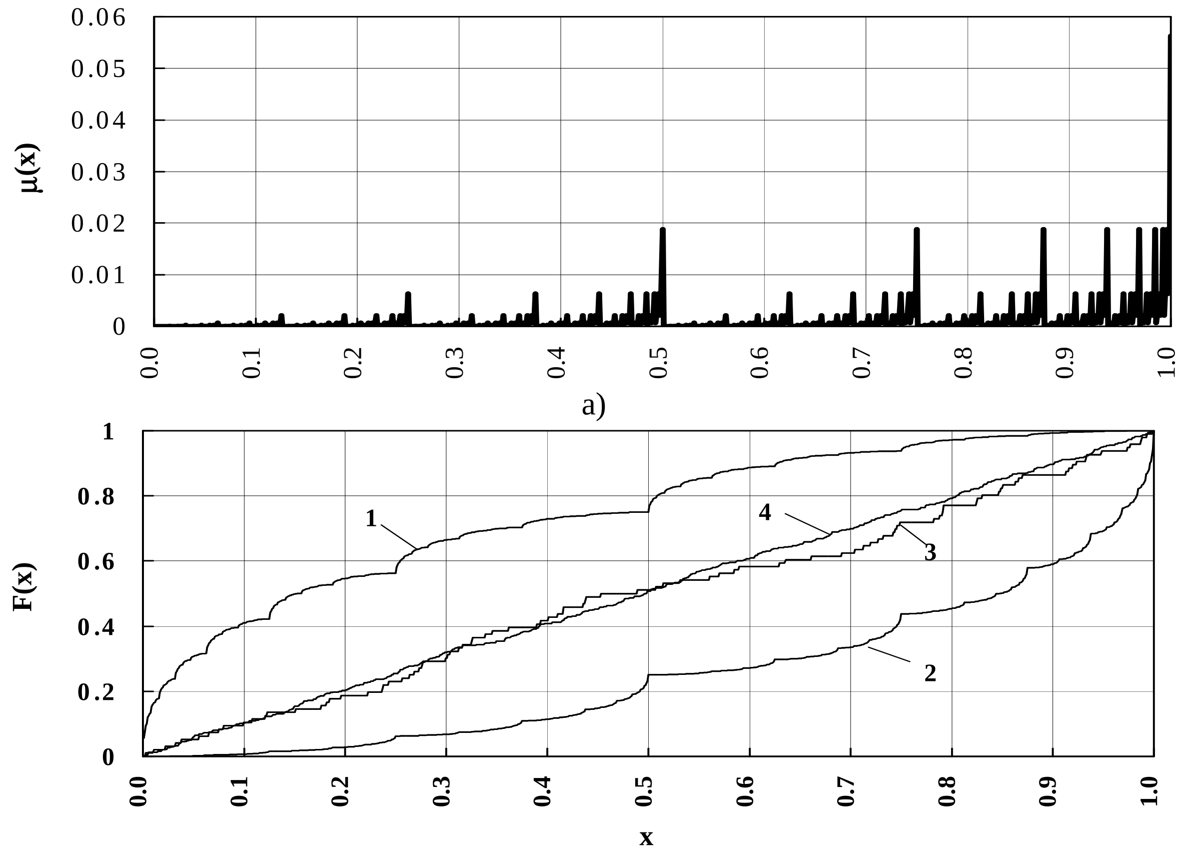
normalized by $|a|^{-1/2}$. In Eq.(2) $Supp\ \mu$ is the support of measure μ. This family of functions (3) formed by the dilations, which are controlled by the positive real number $a \in R+$, and translations which are controlled by the real number $b \in R$, of a single function $\psi(t)$ named the mother wavelet. The dilation parameter a controls the frequency of $\psi_{ab}(t)$. The translation parameter b simply moves the wavelet throughout the domain.

The wavelet transform can be regarded as a mathematical microscope [5,6]. Wavelet analysis is a powerful tool for locating singularities because a singularity of a measure $\mu(t)$ at $t(i)$ produces a cone-like structure in the wavelet transform $W_{ab}\{\mu(t)\}$, pointing towards the point $a=0$, $b=t(i)$. The wavelet transform assists visualization of self-similar or self-affine properties of multifractal objects [5]. In particular, it illustrates the complexity of the multifractal under consideration, revealing the hierarchy that governs the relative positioning of the singularities of a probabilistic measure $\mu(t)$. In the point stochastic process which represents the damage accumulation time history these singularities model the relative positioning of the damage jumps in the course of time.

4. Computer simulation study. For the approbation of the technique, a computer simulation study has been done. As a first step, we have carried out a multiscale analysis of data generated by MBP as a result of the series of Bernoulli trials. The MBP produces shorter and shorter temporal subintervals Δt that contain less and less fractions of the total measure. Finally the process generates a multifractal measure, supported by a generalized Cantor set on the unit temporal interval T = [0, 1]. Fig. 1,a shows the plot of the probability mass function, that is measure □(x) of subinterval, located at x as a result of the Bernoulli trials with parameter $p = 0.25$ after n = 12 generations. Fig. 1,b shows the plot of the cumulative distribution function F(x) for the MBP, that is the measure for the interval [0, x], as a function of x

$$F(x) = \sum_{i=0}^{x} \mu_i \ . \tag{4}$$

The cumulative distribution function for the MBP has an evident feature, that is the self-affinity of the function F(x), so the measure F(x) for the interval [0, x] is scaling in the sense that the left half of line 2 in Fig. 1,b is obtained from the whole, and the right half from the whole when transformed by anisotropic dilations.



b)

Fig. 1. The computer simulation results: a - probability mass function μ(x); b - cumulative distribution function F(x) for MBP generated by Bernoulli trials with *p*=0.25 (line 2); *p*=0.75 (1) and for Poisson process (*p*=0.5) with parameter λ=0.1 (3) and λ=0.5 (4)

Connection between parameter p of the Bernoulli trial and multifractal spectrum is considered in [4]. Three types of the failure-cascading process associated with typical form of cumulative distribution function are considered: process with early failures (when $p > 0.5$), independent process ($p = 0.5$), and process with late failures ($p < 0.5$). We use the forms for describing the damage accumulation process: the first process corresponds to the cyclic softening of material, the second - to a unique damage evolution (Palmgren-Miner's rule) and the third - to the cyclic hardening of material [7].

In order to verify the fractality of data obtained from numerical simulations the CWT of the measure generated by the MBP was carried out by using the WaveLab package [6]. A "Mexican hat" wavelet was used because it provided better visualization [8,9]. The graph of the local maxima lines of the CWT skeleton for measure $\mu(t)$ generated by the MBP with p=0.25 on the generalized Cantor set is shown in Fig. 2. The maxima lines are converging towards the singularities of the measure, and they reproduce its hierarchical structure [5]. The symmetry of graph is broken by the non-uniform measure. The successive forkings occur at different scales. They reveal the multifractal nature of measure. By using the classical example of the Bernoulli trials we have proved the technique and software.

5. Wavelet analysis of datasets. Wavelet analysis is known to be a powerful tool for analyzing fractal attractors. We have applied the CWT to data on fractures obtained from tests and operation. Wavelet transform provides a two-dimensional unfolding of the one-dimensional time history, resolving both the time and the scale as independent variables. The multifractal structures proposed in the damage accumulation process are real-time structures, in contrast to fractal attractors, which reside in phase space. Thus the wavelet analysis can be applied directly to data obtained from tests, operation or from inspections of technical state in operation.

Birnbaum-Saunders et al. data [10] have been used. According to [10,11] the test specimens were 6061-T6 aluminum strips. They were deflected in reverse bending and three stress amplitudes were used: 145 MPa, 179 MPa, 214 MPa - for the first, second and third sample respectively. Specimens were tested to failure. The plots of empirical distribution function $F(N_c)$ as a function of the loading cycles N_c are shown in Fig. 3. The empirical distribution functions of the lifetime data are the examples of non-differentiable functions, they are constant almost everywhere except in those points where failures occur. We use a generalized "devil staircase" fractal [2] for describing the empirical distribution function.

The wavelet analysis of the datasets was carried out by using the WaveLab [6]. Absolute values of the CWT $W_{ab}\{\mu(t)\}$ coefficients and the skeletons were computed with the "Mexican hat" wavelet. Increasing the resolution reveals progressively the successive generations of branching. The symmetry of the plot is broken by non-uniformity of probabilistic measure. Let $N_L(a)$ be the number of local maxima lines in the CWT skeleton at the scale a. The concentration of data points around the straight line observed in the plot of $\log(N_L(a))$ versus $\log(a)$ can be regarded as a quantitative indication of the self-similarity of the event sequence in real data sets [5] (Fig. 4). Failure occurrence is probabilistic process, which results in the formation of self-affine temporal clusters. Wavelet analysis of empirical data on damage accumulation provides probabilistic evidence for the existence of a multiplicative process hidden in the temporal ordering of the damage accumulation jumps sequence.

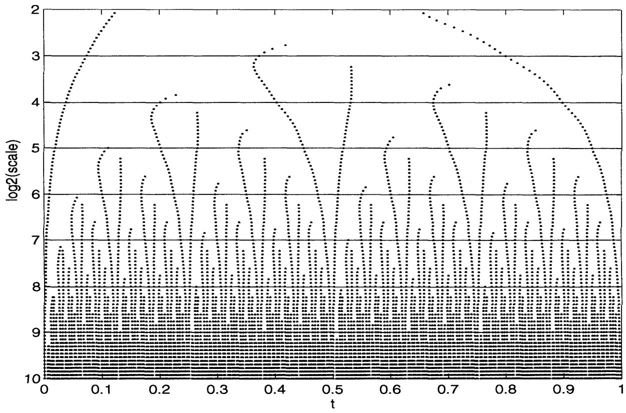


Fig. 2. Local maxima lines of the CWT skeleton for the multifractal measure $\mu(t)$ generated by the MBP with p = 0.25 on the generalized Cantor set

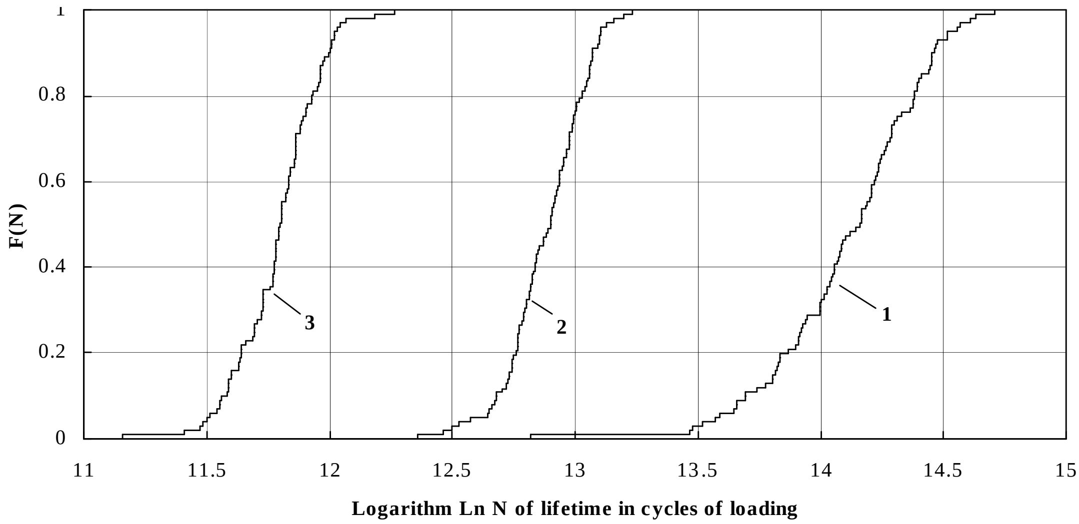


Fig. 3. Empirical distribution functions $F(N_c)$ of lifetime data for the samples 1, 2 and 3 of specimens

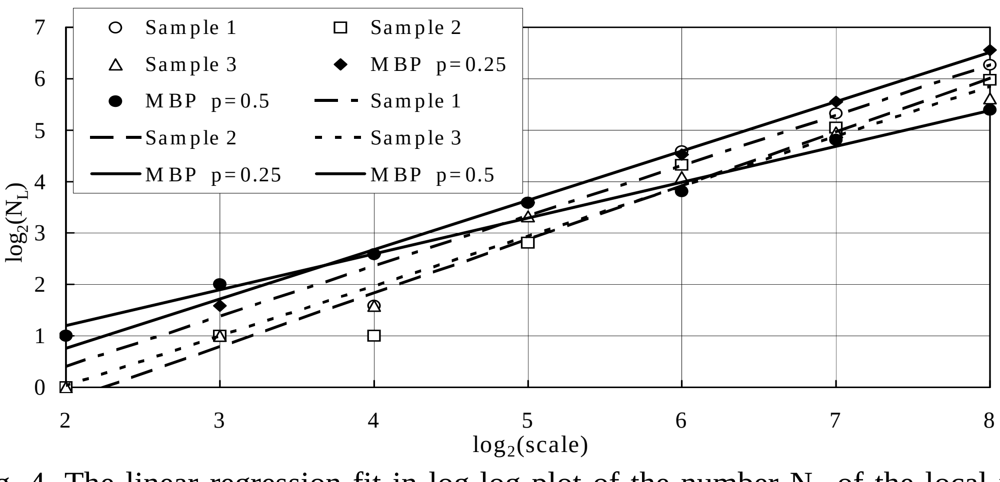


Fig. 4. The linear regression fit in log-log plot of the number N_L of the local maxima lines in the CWT skeleton versus the scale reveals the self-similarity of data generated by MBP and empirical lifetime data for three samples of specimens

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