MATHEMATICAL STUDY OF THE EFFECTS OF APPLIED STRESS, *T*-STRESS AND BACK STRESS IN PHOTOELASTIC FRINGE PATTERNS

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Abstract. This work is an attempt at developing a novel mathematical model to describe the stresses near the crack tip, taking into consideration the effects of plasticity. The focus is on describing how the applied stress normal to the crack, herein referred to as the *K*-stress, *T*-stress and 'back stress' induced by plasticity along the crack flank and in the crack tip plastic zone influence the crack tip elastic stress fields. The important features emerging from this study are that the sign and magnitude of each term can substantially alter the crack tip stress fields, and hence influence the photoelastic fringe patterns. To validate the mathematical model, polycarbonate compact tension specimens have been used and observed in a transmission polariscope in order to study the single effect of a pure 'back stress' (acting as an interfacial shear stress at the elastic-plastic boundary) and combination effects of *K*-stress, *T*-stress and 'back stress'. It is observed that the fringe patterns obtained through experiment show good agreement with those derived by mathematical modelling.

Introduction

The first term in Williams' series expansion [1] for elastic stresses around the mode I crack tip is often expressed in terms of *K*, the stress intensity factor, whereas the second term is known as the *T*-stress. Classical theories of fracture mechanics make the presumption that the near crack tip stresses or strains can be characterized by a single parameter such as *K*. However, extensive studies over the last two decades have shown that the *T*-stress is also an important parameter for describing the states of stresses and strains near the crack tip [2]. More recently, it has been proposed that the plastic enclave around a crack tip and along the crack flanks will shield the crack from the full influence of the applied elastic stress field [3]. Crack tip shielding includes crack flank contact forces (so-called crack closure) and the compatibility-induced interfacial shear stress at the elastic-plastic boundary. Both of these will introduce a 'back stress' on the elastic field. These influences and their variability appear likely to be a major factor underlying the ongoing controversies and uncertainties associated with fatigue crack closure [3].

The main objective in this study is to develop a mathematical model that describes these influences on elastic crack tip stress fields. It builds on work performed previously to link an analytical or mathematical model of a crack undergoing external loading and crack wake contact with full field photoelastic stress patterns [4]. The fundamental problem in such work is to identify the real influence or effect on the applied elastic field of stresses arising from plastic deformation associated with crack growth. It is believed that the roles of the *K*-stress, *T*-stress and 'back stress' on crack tip stress fields have to be included in the mathematical model, so that the effects of the plastic zone and plastic wake on crack closure can be identified and understood better. Experimentally obtained photoelastic fringe patterns are then used as a comparison with those obtained from the mathematical model and to find the magnitudes of the 'back stress' terms.

Mathematical Modelling

The principal stress difference ($\sigma_1 - \sigma_2$) and its direction (θ), that are required for comparison between the theoretical stress field and the photoelastic data, are given by:

$$\sigma_1 - \sigma_2 = \left|\sigma_{22} - \sigma_{11} + 2i\sigma_{12}\right| = \left|2(\bar{z}\phi'' + \psi')\right| \quad (1)$$

$$2\theta = \arg(\sigma_{22} - \sigma_{11} + 2i\sigma_{12}) = \arg(2(\bar{z}\phi'' + \psi')) \quad (2)$$

where ϕ and ψ are complex stress functions, σ_{11}, σ_{12} and σ_{22} are components of stress and overbars denote complex conjugates. In a homogeneous and isotropic linear elastic body containing a two-dimensional crack subject to symmetric (mode I) loading, the leading terms in a series expansion [1] of the stress field near the crack tip are:

$$\sigma_{11} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + T + \mathrm{O}(r^{1/2}) \quad (3)$$

$$\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + \mathrm{O}(r^{1/2}) \quad (4)$$

$$\sigma_{12} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} + \mathrm{O}(r^{1/2}) \quad (5)$$

where K_I is the mode I stress intensity factor and r, θ are co-ordinates in conventional polar systems centered at the crack tip. While the first terms in Eqs. (3-5) are singular; the second term in Eq. (3), often known as the *T*-stress (non-singular, finite and bounded) is a constant stress acting parallel to the crack flanks and independent of the distance from the crack tip. The higher order terms $\mathrm{O}(r^{1/2})$ are negligible near the crack tip. On substitution of Eqs. (3-5) into $\sigma_{22} - \sigma_{11} + 2i\sigma_{12}$, it is found that

$$\sigma_{22} - \sigma_{11} + 2i\,\sigma_{12} = Az^{-1/2} + Bz^{-3/2}\bar{z} + Cz^{0} + Dz^{-1/2}\ln z + Ez^{-3/2}\bar{z}\ln z \quad (6)$$

where

$$A = \frac{K_I}{2\sqrt{2\pi}}, \quad B = -\frac{K_I}{2\sqrt{2\pi}} \quad \text{and} \quad C = -T$$

The co-ordinates of a point in Eq. (6) are expressed as a complex number z where $z = x + iy = re^{i\theta}$ and x, y are the Cartesian co-ordinates of the point, relative to an origin at the centre of the crack. The crack is assumed to be internal through-thickness in an infinite body. The objective of adding the last two natural logarithm terms here is to simulate crack closure contact stresses which provide a contribution to the stress field that is assumed to be proportional to $r^{-1/2}$ behind the crack tip. The interfacial shear stress (at the elastic-plastic boundary) acting parallel to the top and bottom of the crack flank are given by

$$\sigma_{12(Top)} = -\frac{1}{2}(A+B)r^{-1/2}, \qquad \sigma_{12(Bottom)} = \frac{1}{2}(A+B)r^{-1/2} \quad (7)$$

In the elastic stress field, it is assumed that the sum of $A+B$ is equal to zero when there are no interfacial shear stresses acting along the crack flanks. However, if the sum of $A+B$ is not equal to zero, then there should be some nontrivial interfacial shear stresses due to the plastic zone and plastic wake which should simulate the situation in a fatigue cracked specimen. One could, in principle, determine the direction of σ_{12} acting along the crack flank. It follows from Eq. (7) that the signs of σ_{12} on the top and bottom of the crack flanks are different, so that the interfacial shear stresses are acting in the same direction on both crack flanks due to the symmetry. If σ_{12} is negative on the top of the crack flank, both interfacial shear stresses on the top and bottom of the crack flanks are acting towards the crack tip and vice versa. Output from the mathematical model is plotted as difference between principal stresses and hence the fringe patterns are directly analogous with photoelastic data.

Parametric Study

In order to validate the proposed mathematical model, a parametric study was performed that was primarily concerned with studying the individual effect of applied stress, *T*-stress and 'back stress' on fringe patterns. The overall contribution of these stresses is modelled and presented in the next section along with experimental photoelastic fringe data. The scope of work is confined to exploring how different magnitudes of *K*-stress, *T*-stress and 'back stress' influence the stress field around the crack tip. This will give a better understanding of how crack closure is experienced by a growing fatigue crack in terms of influences on the elastic stress field ahead of the crack tip. The mathematical model is constructed using Visual Basic 6.0. The parameters A, B, C, D and E in Eq. (6) used in the parametric study are summarised in Table 1. Fig. 1 shows three plots of fringe patterns with increasing applied elastic stress, or *K*-stress. As expected, it is observed that increasing the applied stress causes an increase in the size and quantity of the fringes.

The effects of *T*-stress on the fringe patterns around the crack can be investigated by varying its sign and magnitude. According to Figs. 1-3, the fringes rotate backward for a positive *T*-stress, show no rotation for a zero *T*-stress and rotate forward for a negative *T*-stress. It is shown that both the backward and forward rotations increase as the magnitudes of the *T*-stresses increase. However, the changes of the forward rotations for negative *T*-stresses are not as significant as backward rotations for positive *T*-stresses. It should be noted that, as the magnitude of *C* increases, the size of the fringe increases for both the positive and negative *T*-stresses. However, the effects of the same magnitude of the positive and negative *T*-stresses on the size of fringes are not identical. The size of a fringe corresponding to a negative *T*-stress is significantly larger than that of a positive *T*-stress for a fixed value of *C*.

The fringes obtained from a pure elastic stress field (Figs. 1-3) form a much sharper 'V' ahead of the crack tip than the equivalent fringes from the combination of applied elastic stress field and interfacial shear stress due to plasticity (Fig. 4). In addition, the fringe loops to either side of the crack for the former case return to the crack tip, whereas those for the latter case return to a point behind the crack tip. These findings accord with those reported by James et al. [3] in which both of these changes were ascribed to the plastic deformation of the material in the immediate vicinity of the crack. It is clear from the Fig. 4 that both of these changes become more apparent with an increase in interfacial shear stress.

It is important to recognise that closure contact stresses are modelled in this study as an asymptotic field behind the crack tip. It is therefore clear that the shape of fringes for the applied stress and closure contact stress are identical. The occurrence of contact at the fracture surface causes a load transmission through the contacting sites. Such behaviour would reduce or shield the role of externally applied stress range. The net interpretation of the results shown in Fig. 5 is that the usual effect of the applied stress (size and quantity of the fringe) decreases as closure contact stress increases.

Experimental Verification

To validate the proposed mathematical model, a photoelastic experimental study was carried out to examine the single effect of pure 'back stress' and combination effects of *K*-stress, *T*-stress and 'back stress'. An annealed polycarbonate compact tension (CT) specimen, w = 90mm in width, h = 67mm in height and d = 2mm in thickness, was subjected to fatigue cycling between 0-2.3 MPam$^{0.5}$ (150N) in an Instron screw-driven testing machine. A fatigue crack of length 12mm was grown from the tip of the notch. Then, the applied stress intensity was increased in 0.1MPam$^{0.5}$ (6.5N) increments up to 1.5MPam$^{0.5}$ (98N). The loading frame was located within a circular polariscope, which was illuminated by a monochromatic light source. A recording was made of the fringe patterns in an area approximately 30mm x 40mm around the crack tip.

The effect of the plastic zone surrounding the crack is clearly visible in the fringe pattern during loading (Fig. 6a). The most obvious observation was that the fringe loops at the crack tip did not disappear even after the far-field load is fully removed from the specimen, showing that residual compressive strain remains in the specimen ahead of a fatigue crack which has previously been subjected to far-field cyclic tension. The largest residual compressive strains exist in the immediate vicinity of the crack tip. It is interesting to observe that the size of the fringes due to residual compressive strain and 'back stress' reduces when a small load is applied, and then increase again as the applied load is increased.

Fig. 6a shows the fringe pattern of the polycarbonate CT specimen observed in a bright field circular polariscope with applied stress intensity of 1.0 $MPam^{0.5}$ (65.4N) whereas Figs. 6b and 6c respectively show the mathematical modelling of applied elastic stress field with and without a plasticity effect. The fringe pattern observed in the polycarbonate CT specimen is considerably influenced by fracture surface contact and interfacial shear stress. It is clear that the fringe pattern in Fig. 6b is a better match to the experimental fringe pattern than that shown in Fig. 6c.

Table 1. Parameters A, B, C, D and E in Eq. (6) used in the parametric study.

Case	Parameter					Applied Stress	*T*-stress	Interfacial Shear Stress	Closure Contact Stress
	A	*B*	*C*	*D*	*E*				
1a	20	-20	0	0	0	Increase	0	0	0
1b	50	-50	0	0	0	Increase	0	0	0
1c	100	-100	0	0	0	Increase	0	0	0
2a	100	-100	-2	0	0	Fixed	Increase (+)	0	0
2b	100	-100	-5	0	0	Fixed	Increase (+)	0	0
2c	100	-100	-8	0	0	Fixed	Increase (+)	0	0
3a	100	-100	2	0	0	Fixed	Increase (-)	0	0
3b	100	-100	5	0	0	Fixed	Increase (-)	0	0
3c	100	-100	8	0	0	Fixed	Increase (-)	0	0
4a	20	-20	-2	0	0	Fixed	Fixed (+)	Increase	0
4b	50	-20	-2	0	0	Fixed	Fixed (+)	Increase	0
4c	80	-20	-2	0	0	Fixed	Fixed (+)	Increase	0
5a	50	-50	-2	0	0	Fixed	Fixed (+)	0	Increase
5b	50	-50	-2	-3	3	Fixed	Fixed (+)	0	Increase
5c	50	-50	-2	-6	6	Fixed	Fixed (+)	0	Increase

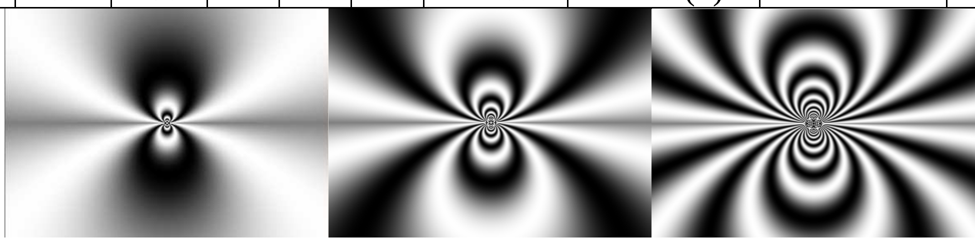

Fig. 1. Fringe patterns with increasing applied elastic stress or *K*-stress (Cases 1a, 1b, 1c from left to right) for a mode I crack growing from left to right.

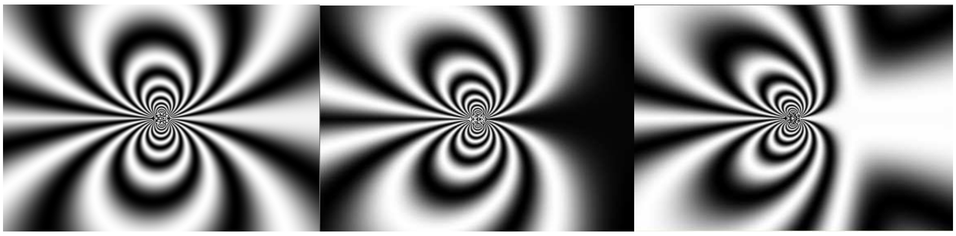

Fig. 2. Fringe patterns with increasing positive *T*-stress (Cases 2a, 2b, 2c from left to right) for a mode I crack growing from left to right.

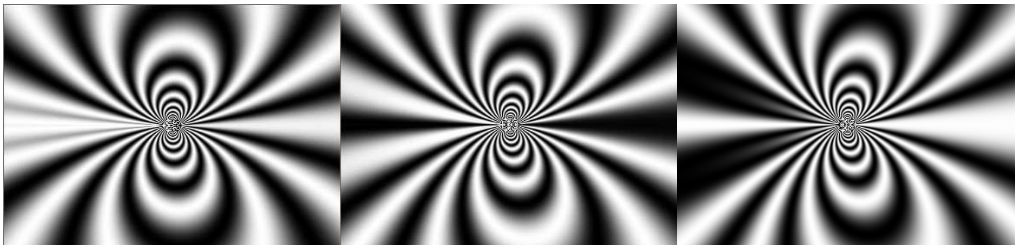

Fig. 3. Fringe patterns with increasing negative *T*-stress (Cases 3a, 3b, 3c from left to right) for a mode I crack growing from left to right.

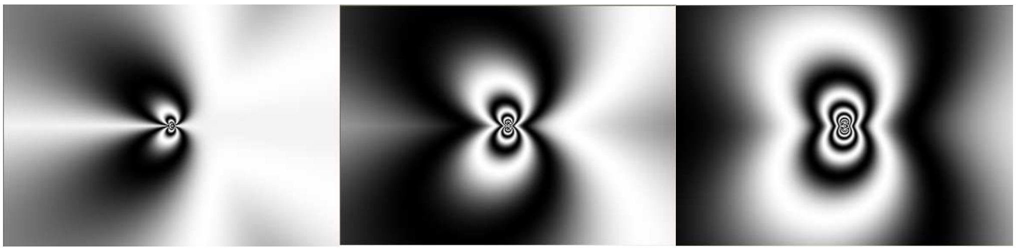

Fig. 4. Fringe patterns with increasing interfacial shear stress (Cases 4a, 4b, 4c from left to right) for a mode I crack growing from left to right.

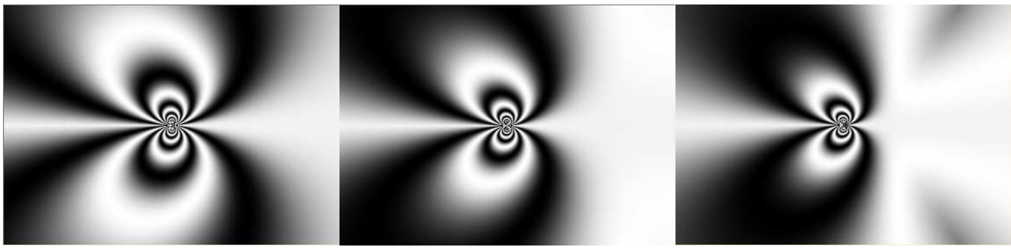

Fig. 5. Fringe patterns with increasing closure contact stress (Cases 5a, 5b, 5c from left to right) for a mode I crack growing from left to right.

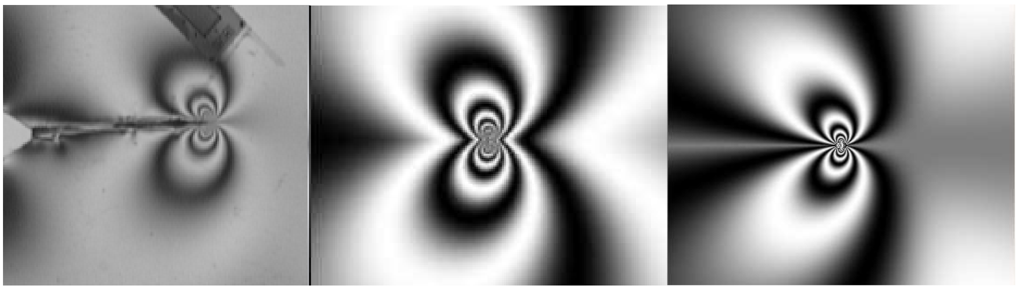

Fig. 6. a) Fringe pattern observed in a bright field circular polariscope, b) Mathematical modelling of applied elastic stress field with and c) without a plasticity effect.

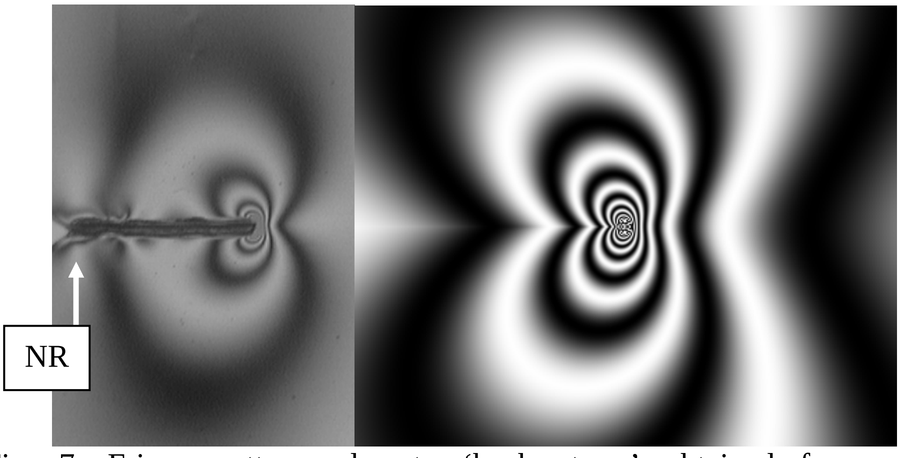

Fig. 7. Fringe patterns due to 'back stress' obtained from a) Photoelasticity, b) Mathematical modelling.

In order to further examine the concept of the interfacial shear stress, a stationary crack of length 10mm was cut into a polycarbonate CT specimen using a jewellers saw 0.15mm in width. A uniform plastic zone in the wake of the crack front was very clearly visible and gives rise to a fringe pattern that is interpreted as arising from compatibility of strain or strain mismatch at the elastic and plastic boundary. Hence the fringe patterns obtained as shown in Fig. 7a are solely due to 'back stress' on the elastic field which may include an interfacial shear stress term and closure contact stress. Again, the fringe loops to either side of the crack emanate and return to a point ahead and behind the crack tip. It is observed that the fringe pattern derived from the mathematical model as shown in Fig. 7b shows good agreement with the one obtained experimentally. However, excessive residual plasticity near the notch root (point NR in Fig. 7a) have a significant effect on the fringe pattern and make the junction of fringe and crack flank a different shape to that seen in the model. There may also be a contribution to this from boundary effects. It is finite geometry in the real specimen, whereas the mathematical model assumes an infinite body.

Conclusions

The main objective of this research was to develop a mathematical model that describes crack tip stress fields in the presence of several components associated with crack growth and specimen geometry. The forces that have been included are applied *K*-stress, *T*-stress and the 'back stress' induced by plasticity. Taking another step further, the solution of the mathematical modelling can be used to explore how different load conditions affect the crack tip stress fields and hence alter the photoelastic fringe patterns. The major conclusions drawn from the present work are as follows:

1. The fringes rotate backward for a positive *T*-stress, show no rotation for a zero *T*-stress and rotate forward for a negative *T*-stress. It is shown that the amount of rotation backwards and forwards of fringe patterns is a function of *T*-stress magnitude. Both the magnitudes of positive and negative *T*-stress affect the size of the fringe pattern.
2. Two primary features can be easily identified in a fringe pattern near a fatigue crack. Firstly, the fringes near a fatigue crack with residual deformation arising from plasticity form a 'U' shape rather than a much sharper 'V' as normally obtained from the pure elastic stress field. Secondly, the fringe loops to either side of a fatigue crack emanate and return to a point ahead and behind the crack tip due to plasticity.
3. The size of the fringes due to residual compressive strain and 'back stress' first reduces when a small load is applied, and then increases subsequently as the applied load increases. As mathematically predicted, the effects of these residual compressive strain and closure contact stress are overcome by the applied tensile stress. These observations form, to the knowledge of the authors, the first stress based visualisation of the phenomenon of crack tip shielding in a real specimen.

References

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